Contents

Chapter 0

Brief Review

0.1 Linear Equations

0.1 Example 1

Most people's favorite version of a linear equation is this:

$$y = mx + b \qquad \text{"slope-intercept form"}$$

where

$m =$ slope (i.e. the ratio of how much the line rises, divided by how much the line goes horizontally),

$b = y$-intercept (i.e. where the line hits the y-axis).

Graph the following lines on the graphs paper below.

(a) $y = 3x + 2$

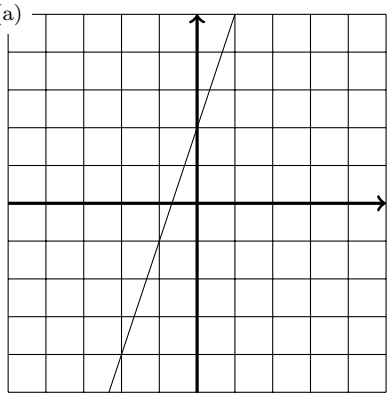
(b) $y = -\frac{1}{2}x - 5$

(c) $y = -\frac{2}{3}x + 5$

(d) $y = 5x - 7$

Solution:

(a)



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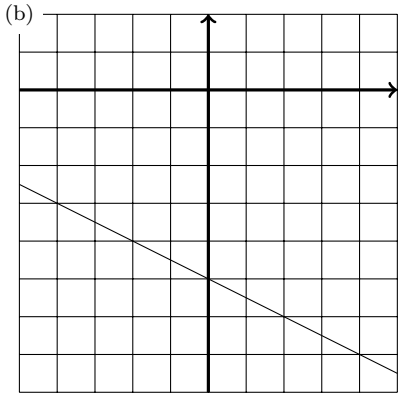

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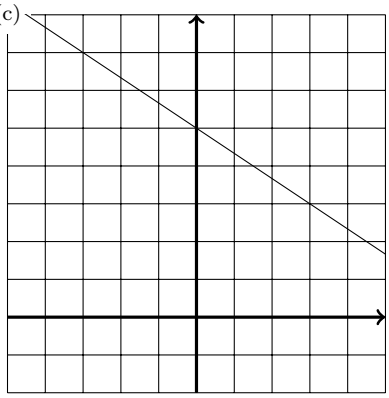

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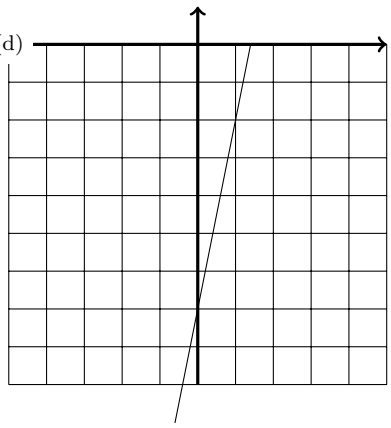


0.1 Example 2

The following equations all define a line, but are not in the usual slope-intercept form, i.e. of the form $y = mx + b$.

Turn the following equations into slope-intercept, and then graph them below

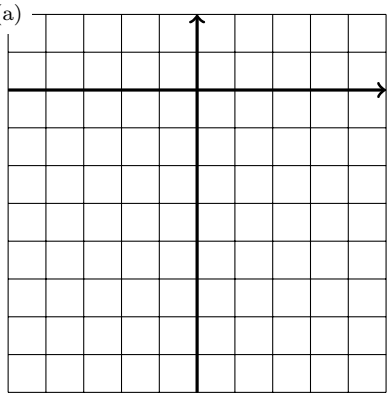
(a) $2y + x = -10$

(b) $3y + 2x = 15$

(c) $y = 5(x + 2) - 17$

(d) $y - 10 = 3x - 8$

(a)



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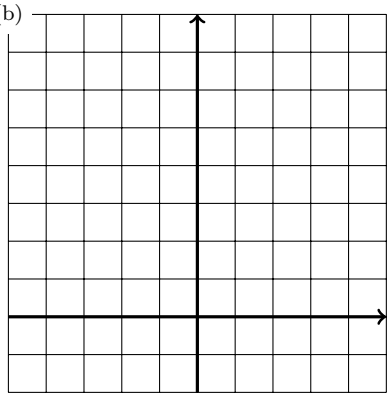

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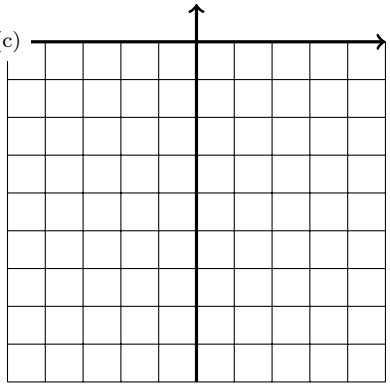

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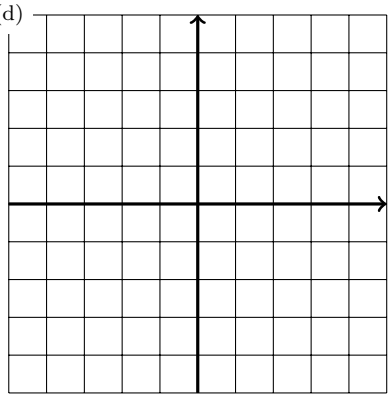


Solution:

(a)

$$\begin{aligned} 2y + x &= -10 \\ 2y &= -x - 10 \\ y &= -\frac{1}{2} - 5 \end{aligned}$$

This is the same as part (b) in the previous example, and so this is graphed above.

(b)

$$\begin{aligned} 3y + 2x &= 15 \\ 3y &= -2x + 15 \\ y &= -\frac{2}{3}x + 5 \end{aligned}$$

This is the same as part (c) in the previous example, and so this is graphed above.

(c)

$$\begin{aligned} y &= 5(x + 2) - 17 \\ y &= 5x + 10 - 17 \\ y &= 5x - 7 \end{aligned}$$

This is the same as part (d) in the previous example, and so this is graphed above.

(d)

$$\begin{aligned} y - 10 &= 3x - 8 \\ y &= 3x - 8 + 10 \\ y &= 3x + 2 \end{aligned}$$

This is the same as part (a) in the previous example, and so this is graphed above.

0.1 Example 3

In some problems the quickest way to write a linear equation is like this

$$y = m(x - x_0) + y_0 \qquad \text{“point-slope form}^* \text{”}$$

where

$$m = \text{a given slope,}$$
$$(x_0, y_0) = \text{a given point.}$$

(a) Find the point-slope form equation of the line through the point $(-2, 3)$ with slope 5.

(b) Turn the equation from (a) into slope-intercept form.

(c) Find the point-slope form equation of the line through the point $(-2, 3)$ with slope $-1/2$.

(d) Turn the equation from (c) into slope-intercept form.

Solution:

(a)

$$y = m(x - x_0) + y_0$$
$$y = 5(x + 2) + 3$$

(b)

$$y = 5(x + 2) + 3$$
$$y = 5x + 10 + 13$$
$$y = 5x + 23$$

(c)

$$y = m(x - x_0) + y_0$$
$$y = -\frac{1}{2}(x + 2) + 3$$

(d)

$$y = -\frac{1}{2}(x + 2) + 3$$

* Sometimes people write point-slope as $y - y_0 = m(x - x_0)$. That's ok, there's more than one way to write it. But the version I've given is more useful because it's written as an explicit function, and in any case it's the version I want you to use.

$$y = -\frac{1}{2}x - \frac{1}{2} \cdot 2 + 3$$
$$y = -\frac{1}{2} - 1 + 3$$
$$y = -\frac{1}{2}x + 2$$

0.1 Example 4

This example is meant to show that sometimes it makes sense to think about a problem using the point-slope form of a line.

Suppose that today my son is 52 inches tall and growing at 1.5 inches per year.

(a) Roughly speaking, how tall will he be tomorrow?

(b) How tall will he be in one year?

(c) How tall will he be in two years?

(d) Write a formula for y (=height) as a function of t (=the calendar year), using point-slope form.

Solution:

(a) Basically, tomorrow he'll be about the same height as today:

$$\text{height tomorrow} \approx 52$$

(b) In one year he will grow roughly another 1.5 inches

$$\text{height in one year} \approx 52 + 1.5$$

(c) In two years he should grow another 1.5 inches twice

$$\text{height in two years} \approx 52 + 1.5(2)$$

(d)

$$\begin{aligned} \text{height in a bunch of years} &= 52 + 1.5 \times (\#\text{ of years }) \\ &= 52 + 1.5 \times (t - 2017) \end{aligned}$$

Note that this is the point-slope equation:

$$\begin{aligned} y &= 52 + 1.5(t - 2017) \\ y &= y_0 + m\,(x - x_0) \end{aligned}$$

0.1 Example 5

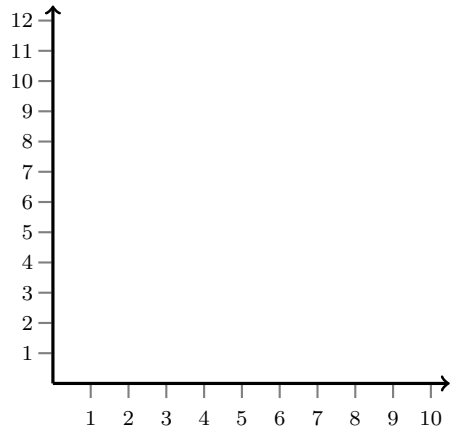
This example is meant to show that it's actually quite easy to graph a line in point-slope form.

(a) On the graph paper below, graph the point $(5, 7)$ with a circle about like this $\bullet$

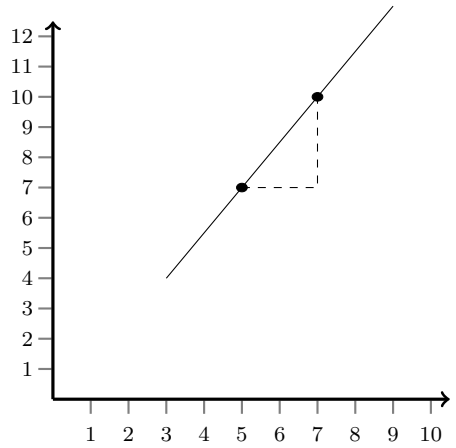
(b) Add to the graph a second large point, $\bullet$ that is 2 places to the right right and 3 places up; mark the distances of 2 and 3 with dashed lines.

(c) Draw a line through the two points you have labeled.

(d) Describe what the graph you made has to do with the line $y = \frac{3}{2}(x-5)+7$.



Solution:



The first lesson is that it was easy to draw this line geometrically: draw one point, count over and up, draw a second point. The second lesson is that we can see this information in the equation $y = \frac{3}{2}(x-5)+7$. The "5" and the "7" is the point we start with. And the slope "$\frac{3}{2}$" is pretty much where we always see it. So really, this shouldn't be any harder than using the $y = mx + b$ equation.

0.2 Fractions

0.2 Example 1

(a) Add the fractions, and simplify if possible: $\frac{5}{14} + \frac{7}{14}$.

(b) Add the fractions, and simplify if possible: $\frac{17}{x} + \frac{3}{x}$.

(c) Get a common denominator and combine the fractions: $\frac{3}{10} + \frac{8}{15}$.

(d) Get a common denominator and combine the fractions: $\frac{3}{7} + \frac{2}{11}$.

(e) Get a common denominator and combine the fractions: $\frac{3}{7} + \frac{x}{11}$.

(f) Get a common denominator and combine the fractions: $\frac{3}{x} + \frac{x}{11}$.

(g) Multiply the fractions, and simplify if possible: $\frac{-5}{3} \cdot \frac{7}{10}$

(h) Multiply the fractions, and simplify if possible: $\frac{x}{2} \cdot \frac{x}{7}$.

(i) Multiply the fractions, and simplify if possible: $\frac{3x}{2} \cdot \frac{-13}{5x}$.

(j) Simplify until you get a single fraction, with no compound fractions: $x\left(\frac{1+\frac{1}{x}}{x+\frac{1}{x}}\right)$

Solution:

(a) $\frac{5+7}{14} = \frac{12}{14} = \frac{6}{7}$.

(b) $\frac{17+3}{x} = \frac{20}{x}$

(c) $\frac{9}{30} + \frac{16}{30} = \frac{25}{30} = \frac{5}{6}$

(d) $\frac{33}{77} + \frac{14}{77} = \frac{47}{77}$

(e) $\frac{33}{77} + \frac{7x}{77} = \frac{33+7x}{77}$

(f) $\frac{33}{11x} + \frac{x^2}{11x} = \frac{33+x^2}{11x}$

(g) There are two ways you can do this. Multiply, then cancel:

$$\frac{-35}{30} = \frac{-35 \div 5}{30 \div 5} = \frac{-7}{6}$$

or you can cancel, then multiply second:

$$\frac{-\cancel{5}^{1}}{3} \cdot \frac{7}{\cancel{10}_{2}} = \frac{-1 \cdot 7}{3 \cdot 2} = \frac{-7}{6}$$

(h) $\dfrac{x}{2} \cdot \dfrac{x}{7} = \dfrac{x^2}{14}$.

(i) As before, you can multiply then cancel:

$$\frac{3x}{2} \cdot \frac{-13}{5x} = \frac{3x(-13)}{2(5x)} = -\frac{-39\cancel{x}}{10\cancel{x}} = -\frac{-39}{10}$$

or you can cancel, then multiply

$$\frac{3\cancel{x}}{2} \cdot \frac{-13}{5\cancel{x}} = \frac{3(-13)}{2(5)} = -\frac{-39}{10}$$

(j) I'll do this one in a fair amount of detail:

$$x\left(\frac{1+\frac{1}{x}}{x+\frac{1}{x}}\right) = \frac{x}{1}\left(\frac{1+\frac{1}{x}}{x+\frac{1}{x}}\right) = \frac{x\left(1+\frac{1}{x}\right)}{x+\frac{1}{x}} = \frac{x+\frac{x}{x}}{x+\frac{1}{x}} = \frac{x+1}{x+\frac{1}{x}}$$

To cancel the last $\dfrac{1}{x}$ we multiply the top and the bottom of this last fraction by x and simplify:

$$\frac{x}{x} \cdot \frac{x+1}{x+\frac{1}{x}} = \frac{x(x+1)}{x\left(x+\frac{1}{x}\right)} = \frac{x(x+1)}{x^2+1}$$

0.3 Exponents

0.3 Example 1

Recall:

$$a^{-b} \text{ means } \frac{1}{a^b} \qquad (a^n)^m = a^{nm} \qquad \frac{a^n}{a^m} = a^{n-m}$$

$$a^{1/b} \text{ means } \sqrt[b]{a} \qquad a^n a^m = a^{n+m} \qquad (ab)^n = a^n b^n$$

Using the above properties, simplify the following.

(a) $(-2)^5$

(b) $\dfrac{x^{17}}{x^{22}}$

(c) $4^{-3/2}$

(d) $\sqrt{36x^4}$

Solution:

(a) $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$

(b) $\dfrac{x^{17}}{x^{22}} = x^{17-22} = x^{-5} = \dfrac{1}{x^5}$

(c) $4^{-3/2} = \dfrac{1}{4^{3/2}} = \dfrac{1}{(\sqrt{4})^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

(d) $\sqrt{36x^4} = \sqrt{36}\sqrt{x^4} = 6x^2$

0.3 Example 2

(a) Simplify the following

$$\frac{-2x^{-4}y^6}{3x^3y^{-3}}$$

so that your final is written using only exponents, no fractions, and each base, 2, 3, x and y, appears only once.

(b) Challenge Problem: Simplify the following

$$\left(\frac{(-2x^{-4}y^6)^{-8}}{(3x^3y^{-3})^{-2}}\right)^{-2}$$

so that your final answer has no fractions, and each base, 2, 3, x and y, appears only once.

Solution:

(a)

$$\begin{aligned}\frac{-2x^{-4}y^6}{3x^3y^{-3}} &= -2x^{-4}y^6 \cdot 3^{-1}x^{-3}y^3 \\ &= -2 \cdot 3^{-1}x^{-7}y^9\end{aligned}$$

(b) There's more than one order you can do this in, and it really doesn't matter too much which way you go. But I think it does help to get some sort of a strategy and try to follow that. For instance, you could say "I'll work from the inside out, and simplify as I go." Or you could say, "I'll work from the outside in, and simplify at the end." But what you probably *shouldn't* say is "I'll randomly combine the inside and the outside, and move everything around until I think of something to do with it."

I'll work from the inside out and simplify as I go:

$$\begin{aligned}\left(\frac{(-2x^{-4}y^6)^{-8}}{(3x^3y^{-3})^{-2}}\right)^{-2} &= \left(\frac{(-2)^{-8}x^{32}y^{-48}}{(3)^{-2}x^{-6}y^6}\right)^{-2} \\ &= \left(2^{-8}3^{-(-2)}x^{32-(-6)}y^{-48-6}\right)^{-2} \\ &= \left(2^{-8}3^2x^{38}y^{-54}\right)^{-2} \\ &= 2^{16}3^{-4}x^{-76}y^{108}\end{aligned}$$

0.4 Square roots

0.4 Example 1

Recall that $(5 \cdot 7)^2 = 5^2 \cdot 7^2$. Since this is true, a similar result holds for square roots: $\sqrt{5 \cdot 7} = \sqrt{5} \cdot \sqrt{7}$.

(a) Simplify the following: $\sqrt{4 \cdot 3}$

(b) Simplify the following: $\sqrt{49x}$ (assume that $x > 0$)

(c) Simplify the following: $\sqrt{7x^2}$ (assume that $x > 0$)

Solution:

(a) $\sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

(b) $\sqrt{49x} = \sqrt{49}\sqrt{x} = 7\sqrt{x}$

(c) $\sqrt{7x^2} = \sqrt{x^2}\sqrt{7} = x\sqrt{7}$

0.5 Grouping and expanding terms

0.5 Example 1

Simplify the following:

$$(3y^3 + 9y^2 - 11y + 8) - (-4y^2 + 10y - 6)$$

Solution:

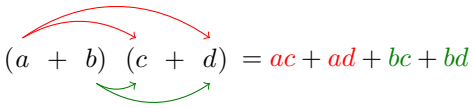
$$\begin{aligned}(3y^3 + 9y^2 - 11y + 8) - (-4y^2 + 10y - 6) &= 3y^3 + 9y^2 - 11y + 8 \ - \ (-4y^2) - 10y - (-6) \\ &3y^3 + 9y^2 - (-4y^2) - 11y - 10y + 8 - (-6) \\ &3y^3 + 9y^2 + 4y^2 - 21y + 8 + 6 \\ &3y^3 + 13y^2 - 21y + 14\end{aligned}$$

0.5 Example 2

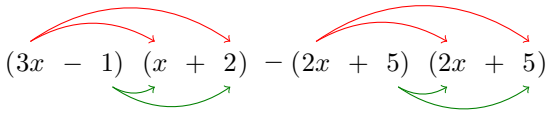
Simplify the following:

$$(3x-1)(x+2)-(2x+5)^2$$

Solution: The main step is FOIL [*] :



We apply FOIL to both $(3x-1)(x+2)$ and to $(2x+5)^2 = (2x+5)(2x+5)$:



$$\begin{aligned} &= [3x^2 + 6x - x - 2] - [4x^2 + 10x + 10x + 25] \\ &= 3x^2 - 4x^2 + 6x - x - 20x - 2 - 25 \\ &= -x^2 - 15x - 27 \end{aligned}$$

[*] Basically "foiling" means you take each thing on the left, and distribute it across the pieces on the right. The letters stand for First Outer Inner Last.

0.6 Quadratics

0.6 Example 1

A quadratic function has the following form

$$y = ax^2 + bx + c$$

Match the following quadratics with their graphs: see if you can do this without using your calculator.

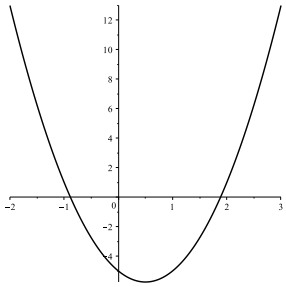
(a) $y = x^2$

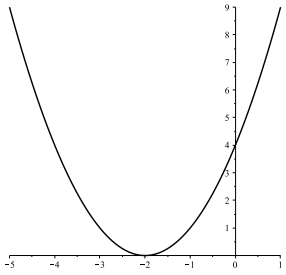
(b) $y = (x + 2)^2$

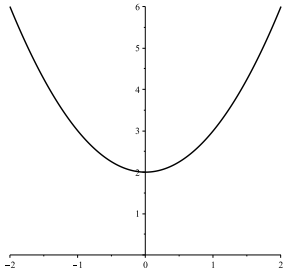
(c) $y = x^2 + 2$

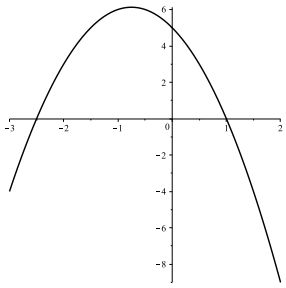
(d) $y = -2x^2 - 3x + 5$

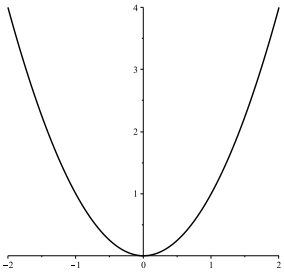
(e) $y = 3x^2 - 3x - 5$





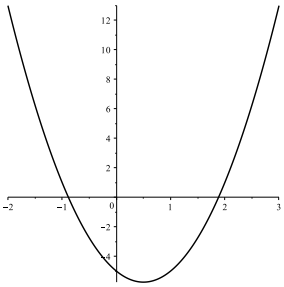




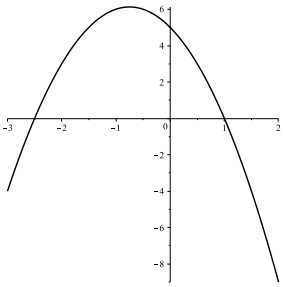


Solution:

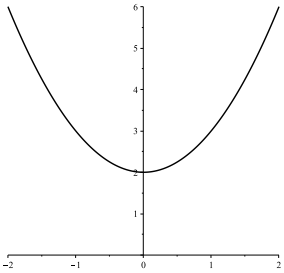
$3x^2 - 3x - 5$



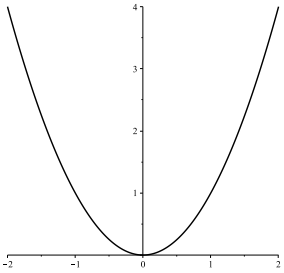
$-2x^2 - 3x + 5$



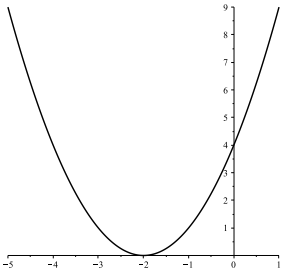
$x^2 + 2$



x^2



$(x + 2)^2$



0.6 Example 2

Factoring a quadratic means to write it as a product. Usually you shouldn't bother to factor a quadratic unless the x^2-coefficient equals 1. In that case, you're trying to write it like this:

$$y = x^2 + bx + c = (x + d)(x + e)$$

There are various tricks in finding d and e, but honestly, in this case, I usually just guess and check as follows: (1) guess two values, d and e, that multiply together to give you c, and then (2) check to see if they add up to b. Note: when I write "$+b$" and "$+c$" and say "add" I'm also including negative numbers in there.

(a) Factor and solve $x^2 + 2x + 1 = 0$.

(b) Factor and solve $x^2 + 3x + 2 = 0$.

(c) Factor and solve $x^2 - 3x + 2 = 0$.

(d) Factor and solve $x^2 - x - 2 = 0$.

(e) Factor and solve $x^2 - x - 12 = 0$.

(f) Factor and solve $x^2 - 8x + 12 = 0$.

Solution:

(a)

$$\begin{aligned} x^2 + 2x + 1 &= 0 \\ (x + 1)(x + 1) &= 0 \\ x &= -1 \end{aligned}$$

(b)

$$\begin{aligned} x^2 + 3x + 2 &= 0 \\ (x + 2)(x + 1) &= 0 \\ x &= -2, -1 \end{aligned}$$

(c)

$$\begin{aligned} x^2 - 3x + 2 &= 0 \\ (x - 2)(x - 1) &= 0 \\ x &= 1, 2 \end{aligned}$$

(d)

$$\begin{aligned} x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x &= -1, 2 \end{aligned}$$

(e)

$$\begin{aligned} x^2 - x - 12 &= 0 \\ (x - 4)(x + 3) &= 0 \\ x &= -3, 4 \end{aligned}$$

(f)

$$\begin{aligned} x^2 - 8x + 12 &= 0 \\ (x - 10)(x + 2) &= 0 \\ x &= -2, 10 \end{aligned}$$

0.6 Example 3

A lot of times it's not worth factoring a quadratic, or it may be impossible. In these cases, just use the quadratic formula

$$ax^2 + bx + c = 0 \quad \Longrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(a) Apply the quadratic formula to $x^2 - x - 12 = 0$.

(b) Apply the quadratic formula to $2x^2 + 3x - 2 = 0$.

Solution:

(a)

$$\begin{aligned} x &= \frac{+1 \pm \sqrt{(-1)^2 - 4(1)(-12)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1 + 48}}{2} \\ &= \frac{1 \pm \sqrt{49}}{2} \\ &= \frac{1 \pm 7}{2} \\ &= \frac{8}{2}, \frac{-6}{2} \\ &= 4, -3 \end{aligned}$$

(Note: this is the same as part (e) in the previous example.)

(b)

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-2)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{9 + 16}}{4} \\ &= \frac{-3 \pm \sqrt{25}}{4} \\ &= \frac{-3 \pm 5}{4} \\ &= \frac{2}{4}, \frac{-8}{4} \\ &= 1/2, -2 \end{aligned}$$

Chapter 1

Functions and Change

1.1 What is a Function?

1.1 Example 1

(a) $f(x) = x^2$, then $f(5) =?$

(b) $g(t) = \sqrt{t^2 + 1}$, $\quad g(0) =?$, $\quad g(-3) =?$

(c) $f(t + 1) =?$

(d) $g(x + h) =?$

Solution:

(a) In the formula "$f(x) = x^2$" we simply replace each "x" with 5 to get

$$f(5) = 5^2$$

Maybe that's the best place to stop, because this problem isn't really about calculating 5^2 it's about how we combine x^2 and 5, which we've done. But I know the curiosity is killing you, so we can go to the next step and say:

$$f(5) = 5^2 = 25$$

(b) With a little practice, you should be able to find $g(0)$ in your head, but for now, let's write out every step:

$$\begin{aligned} g(0) &= \sqrt{0^2 + 1} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

For $g(-3)$ the only real difference is that you should be careful about order of operations with the negative 3. Note that whatever we plug in for t should be squared, including the negative:

$$\begin{aligned} g(-3) &= \sqrt{(-3)^2 + 1} \\ &= \sqrt{9 + 1} \\ &= \sqrt{10} \end{aligned}$$

We'll leave the answer in this form, because the point of this example *isn't* how to find the square root using our calculator. The point is do we know where to put the -3.

(c) For $f(t + 1)$ it might help if we first write f without using x. Remember, "x" is just a letter we use to stand for something else. If you get too fixated on x you might miss how f is really defined, and therefore how to find $f(t + 1)$. Here's a different way to write f:

$$f(\quad) = (\quad)^2$$

Whatever you put into ( ) on the left you should also put into ( )^2 on the right:

$$f(t+1) = (t+1)^2$$

I'll stop there for (c) because the point isn't whether we can FOIL or expand $(t+1)^2$, the point is did we plug $t+1$ in correctly.

(d) As in the last part, let's first write g without using the letter t, just using a blank space which we can plug stuff into:

$$g(\quad) = \sqrt{(\quad)^2 + 1}$$

Now, we should plug $x+h$ into ( ) on both the left and the right:

$$g(x+h) = \sqrt{(x+h)^2 + 1}$$

Again, I won't try to expand and rewrite that any, because it's not the point, and because in this problem it doesn't really help.

1.1 Example 2

The historic Senator Theater is the nearest movie theater to Loyola. Their ticket prices for adults (non-students) seeing a 3D movie are $13.50 for movies after 6 PM, $11 for a matinee (noon – 6 PM), and $9 for an early bird show (before 11 AM). Write a piecewise function $P(t)$ for the price of the ticket where t is the time of the showing.

Solution: Basically we're just taking the given information about pricing and times and arranging it using the notation for piecewise functions. We put the prices first, and then times second. (In general we put formulas first, and the conditions second.) In this case we get

$$P(t) = \begin{cases} 9 & \text{if } t < 11:00 \\ 11 & \text{if } 12:00 \leq t \leq 18:00 \\ 13.50 & \text{if } t > 18:00 \end{cases}$$

GROUP WORK - DISCUSSION BOARD ITEM

Since July 1, 2016, the Maryland minimum wage is \$8.75 per hour (FYI: it will increase to \$9.25 on July 1, 2017). Suppose someone is able to make time and a half per hour of overtime (over 40 hours). If x is the number of hours worked for this week and $f(x)$ is the income function for (gross) income earned that week, answer the following:

(a) $f(x) =?$ in general

(b) $f(A) =?$ for some number $A < 40$

(c) $f(B) =?$ for some number $B > 40$

(d) $f(C) =?$ for some number C where C is NOT an integer and $C < 40$ (for example, $f(10.5) =?$)

(e) $f(D) =?$ for some number D where C is NOT an integer and $D > 40$

Post your questions and answers, even if partial answers to the appropriate discussion board on Moodle! YOU MAY NOT USE THE SAME VALUES FOR A, B, C, AND D as someone else. Comment on others' answers if they made a mistake or you're wondering why they did it that way. Comment if someone's answers could be clarified (be nice!) If several different answers for $f(x)$ have been posted, post which one(s) you think are correct.

Solution: Time and a half rate is $8.75(1.5) = 13.125$

$$I(x) = \begin{cases} 8.75x & \text{if } x \leq 40 \\ 8.75 \cdot 40 + 13.125(x - 40) & \text{if } x > 40 \end{cases}$$

1.1 Example 3

What are the domains of these functions?

(a) $f(x) = 3x - 5$

(b) $g(x) = \sqrt{x+5}$

(c) $h(x) = \dfrac{1}{x}$

(d) $F(t) = \dfrac{\sqrt{5-t}}{t+2}$

Solution: The domain is the set of all numbers that we can plug into our formula. When I say "we can plug in" I mean so that the result is defined, or calculate-able. The way we find the domain is to look for the numbers that are not in it, the numbers that make things undefined or un-calculate-able. In other words, we look for the problem spots.

With algebraic functions like these there are only two possible kinds of problem spots: division by 0, and square roots of negative numbers. These are problems, and so these are not in the domain.

(a) For $3x - 5$ are there any problem spots? Is there any number I could plug in that would lead to division by 0 or the square root of a negative number? No. There is no division, and there is no square root here.

No problem spots means that all numbers are ok. Thus, the domain is all real numbers. We write this in two ways:

interval notation	inequalities
$(-\infty, \infty)$	$\infty < x < \infty$

(b) For $\sqrt{x+5}$ are there are problem spots? Is there any number I could plug in that would lead to division by 0 or the square root of a negative number? No for division, yes for the square root of a negative number.

$$\begin{aligned} \sqrt{x+5} \text{ needs} \\ x + 5 &\geq 0 \\ x &\geq -5 \end{aligned}$$

Again, these are the numbers that are ok, i.e. that are in the domain.

As before, we write this in two ways:

interval notation	inequalities
$(-5, \infty)$	$x \geq -5$

(c) For $\frac{1}{x}$ are there are problem spots? Is there any number I could plug in that would lead to division by 0 or the square root of a negative number? No for square roots, yes for division.

$$\begin{aligned} &\frac{1}{x} \text{ needs} \\ &x \neq 0 \end{aligned}$$

Again, these are the numbers that are ok, i.e. that are in the domain.

As before, we write this in two ways:

interval notation	inequalities
$(-\infty, 0) \cup (0, \infty)$	$x \neq 0$

(d) For $\frac{\sqrt{5-t}}{t+2}$ are there are problem spots? Is there any number I could plug in that would lead to division by 0 or the square root of a negative number? Yes for both!

$$\begin{aligned} \sqrt{5-t} &\text{ needs} \\ 5 - t &\geq 0 \\ 5 &\geq t \\ t &\leq 5 \\ \frac{\text{something}}{t+2} &\text{ needs} \\ t + 2 &\neq 0 \\ t &\neq -2 \end{aligned}$$

Again, these are the numbers that are ok, i.e. that are in the domain.

We need both of these conditions to be true: $t \leq 5$ and $t \neq -2$. This is a good way to write the answer, but if we want to make it look like the other answers, then we can write this in two ways:

interval notation	inequalities
$(-\infty, -2) \cup (-2, 5)$	$t < -2$ or $-2 < t \leq 5$

1.2 Linear Functions

1.2 Example 1

For the two points $(1, 2)$ and $(-5, 4)$ what is the slope of the line connecting them?

Solution: There are probably three useful ways to think about the slope

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\Delta y}{\Delta x} \end{aligned}$$

Note that all three of these formulas say the same thing: "rise" really means $y_2 - y_1$, i.e. how much did y change. Similarly, "Δy" is just shorthand notation for "change of y".

The main things you have to watch out for in this kind of problem are: (1) make sure you put the y's on top, (2) make sure you use the numbers in the same order on the top and bottom. I'll illustrate this by keeping the points in the right order, and by underlining things in color:

In the fraction below, these numbers will come first	In the fraction below, these numbers will come second
$(\underline{-5}, \underline{4})$	$(\underline{1}, \underline{2})$

$$\begin{aligned} m &= \frac{\underline{4} - \underline{2}}{\underline{-5} - \underline{1}} \\ &= \frac{\underline{2}}{\underline{-6}} \\ &= -\frac{1}{3} \end{aligned}$$

1.2 Example 2

Find an equation of the line that passes through $(1, 2)$ and $(-5, 4)$.

Solution: We'll start with the point-slope form of the equation of a line:

$$y = m(x - x_0) + y_0$$

From Example 1, we know the slope already, $m = -\frac{1}{3}$. Then we can use either $(1, 2)$ or $(-5, 4)$ for the known point. We'll use $(1, 2)$.

$$y = -\frac{1}{3}(x - 1) + 2$$

There's nothing wrong with leaving the equation like this, but most of us are more used to the slope-intercept form, so we can distribute the numbers and simplify:

$$\begin{aligned} y &= -\frac{1}{3}x + \frac{1}{3} + 2 \\ y &= -\frac{1}{3}x + \frac{1}{3} + \frac{6}{3} \\ y &= -\frac{1}{3}x + \frac{7}{3} \end{aligned}$$

1.2 Example 3

A cab company has an initial charge of \$4.00 plus \$2.20 per mile. Find a formula for the cab fare, C, in dollars, as a function of the number of miles, m.

Solution: There are three things to identify here: where do we put the 4.00, where do we put the 2.20, and which variables should we use?

The 4.00 is the initial charge. The word "initial" means at the very beginning, in this case, before we start driving at all. This means it's the value when we've gone 0 miles, which is another way to say it's the y-intercept, or the vertical intercept.

The 2.20 is a charge per mile. The word "per" is a clue that this is the slope: slope is a ratio, and "per" always means ratio too.

Putting these together we should have

$$\text{something like } y = 2.20 \times \text{something like } x + 4.00$$

The correct somethings are C instead of y and m instead of x. I know it looks weird, but "m" is our variable here, not slope (since m looks so weird, that's why I wrote it out with words first, to make sure I got things in the right place).

$$C = 2.20m + 4$$

1.2 Example 4

ACME company has seen a decline in sales of their product. In 2010 they sold 28.4 million, while in 2016 they sold 22.7 million.

(a) Find a formula for annual sales S, in millions of items, as a linear function of the years t, since 2010.

(b) Predict the sales in 2019.

Solution:

(a) We have two points of the line: $(0, 28.4)$, and $(6, 22.7)$, which gives us

$$m = \frac{22.7 - 28.4}{6 - 0} = \frac{-5.7}{6} = -0.95$$

Luckily, we were given the y-intercept of 28.4 so we have

$$S(t) = -0.95t + 28.4$$

(b) Use $t = 9$ to get

$$S(9) = -0.95(9) + 28.4 = 19.85$$

Thus we predict sales of 19.85 million in the year 2019.

1.3 Rates of change

1.3 Example 1: Problem 12

Table 1.14 shows world bicycle production. 30www.earth-policy.org/Indicators/indicator11_data1.html, accessed April 19, 2005.

Table 1.14 *World bicycle production, in millions*

Year	1950	1960	1970	1980	1990	2000
Bicycles	11	20	36	62	92	101

(a) Find the change in bicycle production between 1950 and 2000. Give units.

(b) Find the average rate of change in bicycle production between 1950 and 2000. Give units and interpret your answer in terms of bicycle production.

Solution:

(a) We simply subtract the bicycle production during one year from the production during another year:

$$101 - 11 = 90$$

The units are "million bicycles", so the full answer is 90 million bicycles.

(b) The average rate of change is, by definition,

$$\frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}}$$

In this case that means

$$\frac{90}{50} = \frac{9}{5} = \frac{18}{10} = 1.8 \, {}^{\text{Million Bicycles}}/_{\text{Year}}$$

where the units are million bicycles per year.

The interpretation is this: from 1950 to 2000, production of bicycles increased on average 1.8 million bicycles per year.

1.3 Example 2

Find the average rate of change of $f(x) = 4x^2 - 2$ between $x = -1$ and $x = 3$.

Solution: The average rate of change is, by definition,

$$\frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}}$$

In this case that means

$$\begin{aligned}\frac{f(3) - f(-1)}{3 - (-1)} &= \frac{4(9) - 2 - (4(1) - 2)}{4} \\ &= \frac{36 - 2 - 4 + 2}{4} \\ &= \frac{32}{4} \\ &= \boxed{8}\end{aligned}$$

1.3 Example 3: Problem 46*

Consider two situations: (1) A company has an increase in sales from \$100,000 to \$500,000; (2) A company has an increase in sales from \$20,000,000 to \$20,500,000.

(a) Which absolute change is bigger?

(b) Which relative change is bigger? Justify your answer.

Solution:

(a) The change is situation (1) is $400,000$ and the change in situation (2) is $500,000$. Situation (2) is bigger.

(b) The relative change is situation (1) is

$$\begin{aligned}\frac{\Delta y}{y_0} &= \frac{400000}{100000} \\ &= 400\%\end{aligned}$$

The relative change in situation (2) is

$$\begin{aligned}\frac{\Delta y}{y_0} &= \frac{500000}{20000000} \\ &= \frac{5}{200} \\ &= \frac{1}{40} \\ &= 2.5\%\end{aligned}$$

The relative change in situation (1) is bigger.

1.4 Applications of Functions to Economics

1.4 Example 1: Problem 20

A company producing jigsaw puzzles has fixed costs of \$6000 and variable costs of \$2 per puzzle. The company sells the puzzles for \$5 each.

(a) Find formulas for the cost function, the revenue function, and the profit function.

(b) Sketch a graph of $R(q)$ and $C(q)$ on the same axes. What is the break-even point, q_0, for the company?

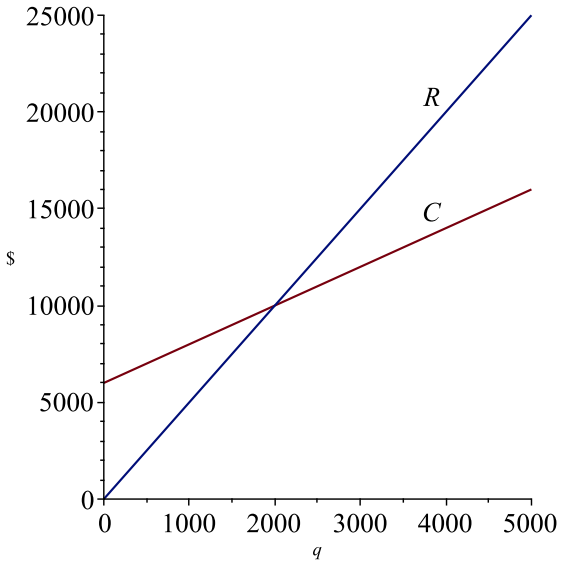
(c) What is the marginal cost?

Solution:

(a)

$$\begin{aligned} C(q) &= \text{fixed cost} + \text{variable cost} \times \text{number of puzzles} \\ &\quad 6000 + 2q \\ R(q) &= \text{sale price} \times \text{number of puzzles} \\ &= 5q, \\ P(q) &= \text{revenue} - \text{costs} \\ &= 5q - (6000 + 2q) \\ &= 3q - 6000 \end{aligned}$$

(b) The graph is shown below



It appears from the graph that the break-even point is about $q = 2000$, and this is easy to verify algebraically:

$$\begin{aligned} C &= R \\ 6000 + 2q &= 5q \\ 6000 &= 3q \\ q &= 2000 \quad \checkmark \end{aligned}$$

(c) The marginal cost is \$2 per puzzle, the cost of making one additional puzzle.

1.4 Example 2: Problem 32

The demand curve for a product is given by $q = 120,000 - 500p$ and the supply curve is given by $q = 1000p$ for $0 \leq q \leq 120,000$, where price is in dollars.

(a) At a price of \$100, what quantity are consumers willing to buy and what quantity are producers willing to supply? Will the market push prices up or down?

(b) Find the equilibrium price and quantity. Does your answer to part (a) support the observation that market forces tend to push prices closer to the equilibrium price?

Solution:

(a)

$$\begin{aligned}
&\text{quantity consumers are willing to buy} \\
&\qquad = \text{demand} \\
&\qquad = 120,000 - 500(100) \\
&\qquad = 70,000 \\
&\text{quantity producers are willing to make} \\
&\qquad = \text{supply} \\
&\qquad = 1000(100) \\
&\qquad = 100,000
\end{aligned}$$

At a price of \$100, the supply is larger than the demand, so some goods remain unsold and we expect the market to push prices down.

(b)

$$\begin{aligned}
\text{equilibrium is where supply} &= \text{demand} \\
120000 - 500p &= 1000p \\
120000 &= 1500p \\
p = \frac{1200}{15} &= \frac{400}{5} = 80
\end{aligned}$$

The equilibrium price is \$80, and the equilibrium quantity is $80,000$.

The market will push prices downward from \$100, toward the equilibrium price of \$80. This agrees with the conclusion to part (a) which says that prices will drop.

1.5 Exponential Functions

1.5 Example 1: Problem 6

The gross domestic product, G, of Switzerland was 310 billion dollars in 2007. Give a formula for G (in billions of dollars) t years after 2007 if G increases by

(a) 3% per year

(b) 8 billion dollars per year

Solution:

(a)

$$G = P_0 a^t, \text{ where } a = 1 + r$$
$$a = 1 + 0.03 = 1.03$$
$$\boxed{G = 310(1.03)^t}$$

(b) This describes a *constant* rate of change, so it is linear.

$$G = mt + b$$
$$m = 8$$
$$b = 310$$
$$\boxed{G = 8t + 310}$$

1.5 Example 2: modeling Loyola's Tuition part I

For school year 2013–2014, the annual tuition at Loyola University Maryland was \$41,850. For school year 2016–2017, the annual tuition at Loyola was \$45,030. Over this time the tuition grew exponentially with an annual percentage rate of growth of 2.47%.

Assuming that the tuition continues to grow at the same rate, what will it be for the 2019–2020 school year?

Solution: We model the tuition with the following equation

$$T = 41850(1.0247)^t$$

where t is the number of years after 2013.

For 2019 we have $t = 6$ and so tuition is estimated to be

$$T = 41850(1.0247)^6 \approx \$48,448.$$

1.5 Example 3: modeling Loyola's Tuition part II

For school year 2013–2014, the annual tuition at Loyola University Maryland was \$41,850. For school year 2016–2017, the annual tuition at Loyola was \$45,030.

Find r, the relative growth rate, so that this growth is modeled by an exponential equation.

Solution: We want tuition to fit into the formula

$$P = P_0(1+r)^t.$$

We plug the data into this equation:

$$\begin{aligned}
P_0 &= \text{tuition in 2013} \\
&= 41850 \\
P &= \text{tuition in 2016} \\
&= 45030 \\
t &= \text{the number of years gone by} \\
&= 3 \\
45030 &= 41850(1+r)^3 \\
\frac{45030}{41850} &= (1+r)^3 \\
\left(\frac{45030}{41850}\right)^{1/3} &= 1+r \\
r &= \left(\frac{45030}{41850}\right)^{1/3} - 1 \\
&\approx 0.02471 \\
&\approx 2.47\%
\end{aligned}$$

1.6 Natural Logarithm

1.6 Example 1: Problem 2, Solving an Exponential Equation

Solve

$$10 = 2^t$$

using natural log.

Solution: We can't solve this problem all the way in our heads, but you should be able to make a guess that the solution will be between 3 and 4. Why? We know $2^3 = 8$ and $2^4 = 16$, so t is somewhere between 3 and 4.

Here's how we solve it algebraically:

$$\begin{aligned} 10 &= 2^t \\ \ln(10) &= \ln(2^t) \\ \ln(10) &= t\ln(2) \\ t &= \frac{\ln(10)}{\ln(2)} \\ &\approx 3.3219 \end{aligned}$$

1.6 Example 2: modeling Loyola tuition part III

In the school year 2013–2014, the annual tuition at Loyola University Maryland was \$41,850. Since then it has had an annual growth rate of $r = 2.47\%$. Assuming this growth rate continues, when will the tuition reach \$52,000?

Solution: We plug this data into the model

$$P = P_0(1 + r)^t$$

and solve for t:

$$\begin{aligned} 52000 &= 41850(1.0247)^t \\ \frac{52000}{41850} &= 1.0247^t \\ \ln(52000/41850) &= \ln\left(1.0247^t\right) \\ \ln(52000/41850) &= t\ln(1.0247) \\ t &= \frac{\ln(52000/41850)}{\ln(1.0247)} \\ &\approx 8.8997102 \end{aligned}$$

So, the actual year should be about 2022.

1.6 Example 3: Continuous Growth

A city's population starts at 600,000 in 2010 and has a continuous growth rate of 5%. What is the population size in 2017?

Solution: We model this city with

$$P = 600000(1 + 0.05)^t$$

and plug in $t = 7$:

$$\begin{aligned} P &\approx 600000(1.05)^7 \\ &= 844260 \end{aligned}$$

1.7 Exponential Growth and Decay

1.7 Example 1: Problem 14*

A population, currently 200, is growing at 5% per year.

(a) Write a formula for the population, P, as a function of time, t, years in the future.

(b) Graph P against t.

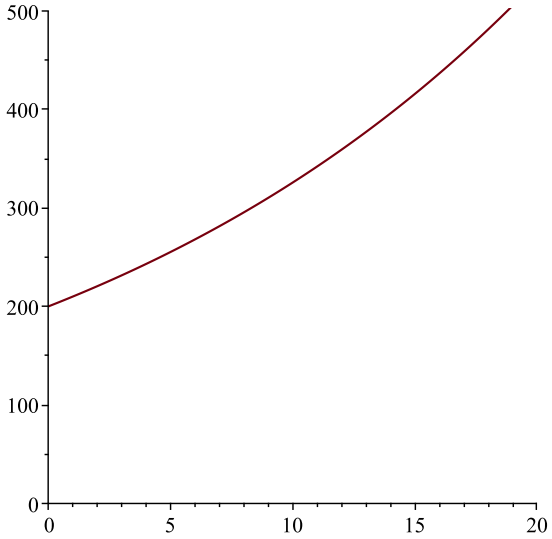
(c) Estimate the population 10 years from now.

(d*) Find the doubling time of the population algebraically.

(e*) Model the same population using a continuous growth rate, compare the graph of this model with the graph from part (b).

Solution:

(a) $P = 200(1.05)^t$



(b)

(c)

$$P(10) = 200(1.05)^{10}$$
$$\approx 325.7789$$

about 326

(d) We solve for when $P = 400$:

$$\begin{aligned} 400 &= 200(1.05)^t \\ 2 &= 1.05^t \\ t &= \ln(2)/\ln(1.05) \\ &\approx 14.2067 \end{aligned}$$

(e) They are asking to change this model to the natural exponential (base e, or Pe^{kt} model. What is the k in this instance?

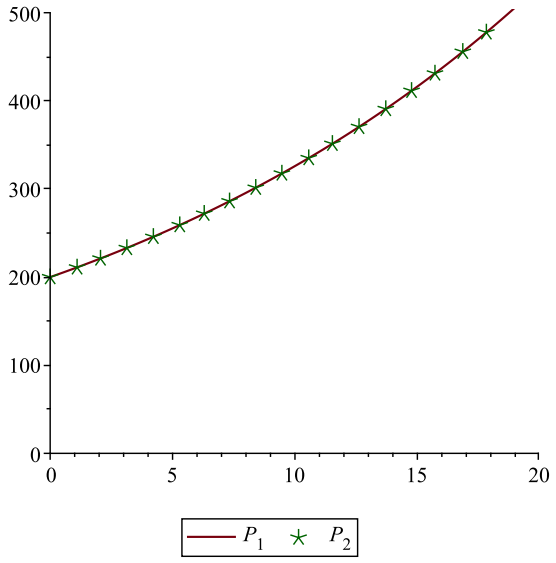
$$200(1.05)^t = 200e^{kt}$$

We can use $t = 1$(or any value of t but $t = 1$ is easiest)

$$\begin{aligned} 1.05 &= e^k \\ \ln(1.05) &= k\ln(e) = k \\ k &\approx 0.04879 \end{aligned}$$

Thus the continuous growth rate is about 4.879% for an annual growth rate of 5%.

We show below the original function, P_1, plotted together with the function $P_2 = e^{0.04879t}$, marked with a few little stars, $\star$. As you can see, the original function and the new one look identical:



1.7 Example 2: Present and Future Value

(a) Find the future value in 8 years of a \$7,000 payment today, if the interest rate is 3.5% compounded continuously.

(b) Find the present value of a \$7,000 payment that will be made 8 years from now if the interest rate is 3.5% compounded continuously.

Solution:

(a) We'll use the equation

$$FV = PVe^{rt}$$

where $PV = 7000$, $r = 0.035$, and $t = 8$:

$$FV = 7000e^{0.035(8)} \approx \$9261.91$$

(b) We'll use the equation

$$FV = PVe^{rt}$$

where $FV = 7000$, $r = 0.035$, and $t = 8$:

$$\begin{aligned} 7000 &= PVe^{.035(8)} \\ PV &= \frac{7000}{e^{.035(8)}} \\ &= 7000e^{-0.035(8)} \\ &\approx \$5290.49 \end{aligned}$$

Chapter 2

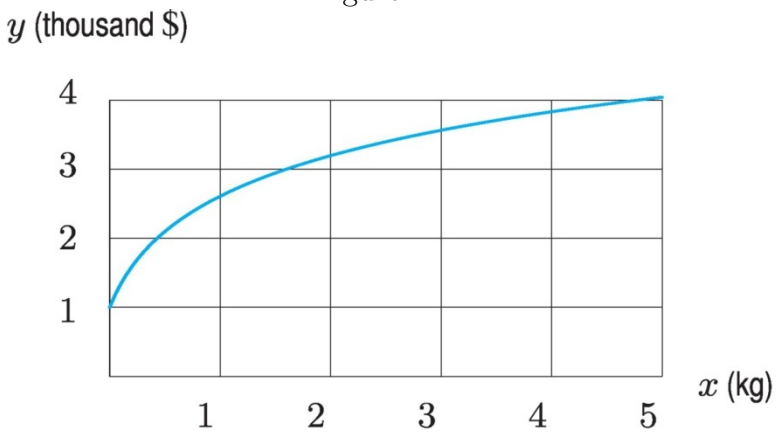
The Derivative

2.1 Tangent and Velocity Problems

2.1 Example 1: Problem 5

Figure 2.12 shows the cost, $y = f(x)$, of manufacturing x kilograms of a chemical.

Figure 2.12



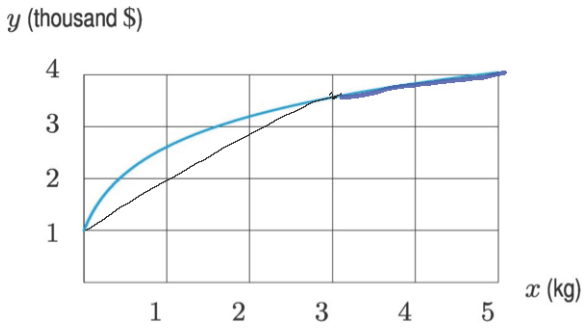
(a) Is the average rate of change of the cost greater between $x = 0$ and $x = 3$, or between $x = 3$ and $x = 5$? Explain your answer graphically.

(b) Is the instantaneous rate of change of the cost of producing x kilograms greater at $x = 1$ or at $x = 4$? Explain your answer graphically.

(c) What are the units of these rates of change?

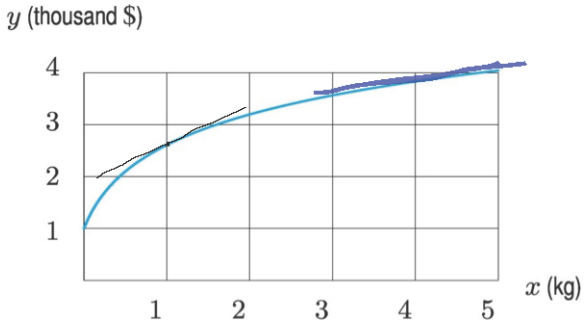
Solution:

(a) In the graph below I've marked, by hand, two line segments on the curve, i.e. two secant lines. One goes from $x = 0$ to $x = 3$, and the other goes from $x = 3$ to $x = 5$



To compare the average rates of change, you should look at the slope of each line segment. Since the one from $x = 0$ to $x = 3$ is steeper, it has a greater slope, and therefore the average rate of change is greater from $x = 0$ to $x = 3$.

(b) In the graph below I've marked, by hand, two tangent lines. One is tangent at $x = 1$ and the other is tangent at $x = 4$.



To compare the instantaneous rates of change, you should look at the slope of the tangent lines. Since the one at $x = 1$ is steeper, it has a greater slope, and therefore the instantaneous rate of change is greater at $x = 1$.

(c) The units are, as always,

$$\frac{\text{units of } y}{\text{units of } x}$$

In this case that's

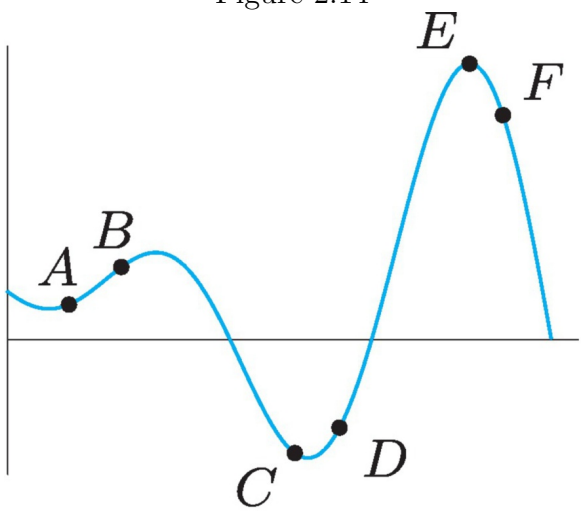
thousands of dollars per kilogram

2.1 Example 2: Problem 12

Match the points labeled on the curve in Figure 2.14 with the given slopes.

Slope	Point
-3	
-1	
0	
$1/2$	
1	
2	

Figure 2.14



Solution: We'll justify below the following answers:

Slope	Point
-3	F
-1	C
0	E
$1/2$	A
1	B
2	D

Let's start by thinking about which point has negative slope: only two points are marked where the graph is going down: C and F. Which one is going down more steeply? You should be able to see that it's F. So, given the choice of two

negative slopes in the table, -3 and -1, we should choose F to have the slope that's more negative: $F \to -3$. Then we must have $C \to -1$.

Now let's think about which point has 0 slope: this should be where the tangent line is perfectly horizontal. There's only one point where this happens: E.

Finally, let's look at three remaining points, where the slope is positive. We can order them by how steep the graph is: It's steepest at D, then next steepest at B, and least steep at A. This means that $D \to 2$, $B \to 1$, and $A \to 1/2$.

2.1 Example 3: The Penny Drop

Suppose we drop a penny from the roof of a very tall building. Then the distance fallen is given by

$$s(t) = 4.9t^2,$$

where s is measured in meters, and t is the number of seconds since the penny has been dropped.

(a) Find the average velocity from $t = 0$ to $t = 7$.

(b) Estimate the instantaneous velocity at $t = 7$.

Solution:

(a)

$$\begin{aligned}\text{average velocity} &= \frac{\text{distance traveled}}{\text{time elapsed}} \\ &= \frac{s(7) - s(0)}{7 - 0} \\ &= \frac{4.9(7^2) - 0}{7} \\ &= 34.3\,\text{m/s}\end{aligned}$$

(b) To estimate the instantaneous velocity we find the average velocity over shorter and shorter time intervals around $t = 7$ seconds. In other words, we will calculate this quantity

$$\frac{s(t) - s(7)}{t - 7} = \frac{4.9(t^2) - 4.9(7^2)}{t - 7}$$

for values of t that are close to 7.

The best way to get a range of values like this is to make a table, either in your calculator, or in a spreadsheet.

t-value (the one not equal to 7)	average velocity from t to 7 $= \dfrac{s(t) - s(7)}{t - 7}$
6	63.7
6.9	68.11
6.99	68.551
6.999	68.595
7.001	68.605
7.1	69.09

Once we have this information, we should look just around the rows that have t as close to 7 as possible, in this case, those are the rows with 6.999 and 7.001. In those rows, the velocity is very close to 68.6, and so that's our guess:

Instantaneous velocity at $t = 7$ is approximately: $68.6\,\text{m/s}$

2.1 Example 4: Estimate the derivative

Using a calculator or an equivalent app, estimate $f'(1)$ for $f(x) = 3x^2$.

Solution: By definition we have

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}.$$

The way to estimate this is to calculate $\dfrac{f(x) - f(1)}{x - 1}$ for more than one value of x, using values that are close to 1.

The best way to get a range of values like this is to make a table, either in your calculator, or in something like `www.desmos.com`:

x-value (the one not equal to 1)	average velocity from x to 1 $= \dfrac{f(x) - f(1)}{x - 1}$
0.5	4.500 $\left(= \dfrac{3(0.5)^2 - 3(1)^2}{0.5 - 1}\right)$
0.9	5.700
0.99	5.970
0.999	5.997
1.001	6.003
1.01	6.030
1.1	6.300
1.5	7.500

Once we have this information, we should look just around the rows that have x as close to 1 as possible, in this case, those are the rows with 0.999 and 1.001. In those rows, the difference quotient is very close to 6, and so that's our guess:

$$f'(1) \approx 6$$

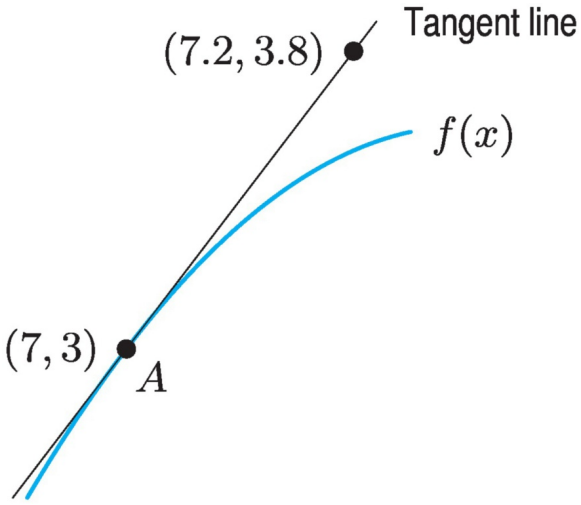
2.1 Example 5: Problem 18

Use Figure 2.16 to fill in the blanks in the following statements about the function f at point A.

(a) $f(\_\_\_\_\_) = \_\_\_\_\_$

(b) $f'(\_\_\_\_\_) = \_\_\_\_\_$

Figure 2.16



Solution:

(a) When we write something like "$f(1) = 2$" we mean that when $x = 1$ we have $y = 2$. Based on the coordinates of point A, we have

$$\boxed{f(7) = 3}$$

(b) When we write something like $f'(10) = 11$ we mean that tangent line at $x = 10$ has slope of 11. Based on the graph we can figure out $f'(7)$ by calculating the slope of the tangent line.

Using the coordinate of the other point on the tangent line, we get the slope of the tangent line is

$$m = \frac{3.8 - 3}{7.2 - 7} = \frac{0.8}{0.2} = 4$$

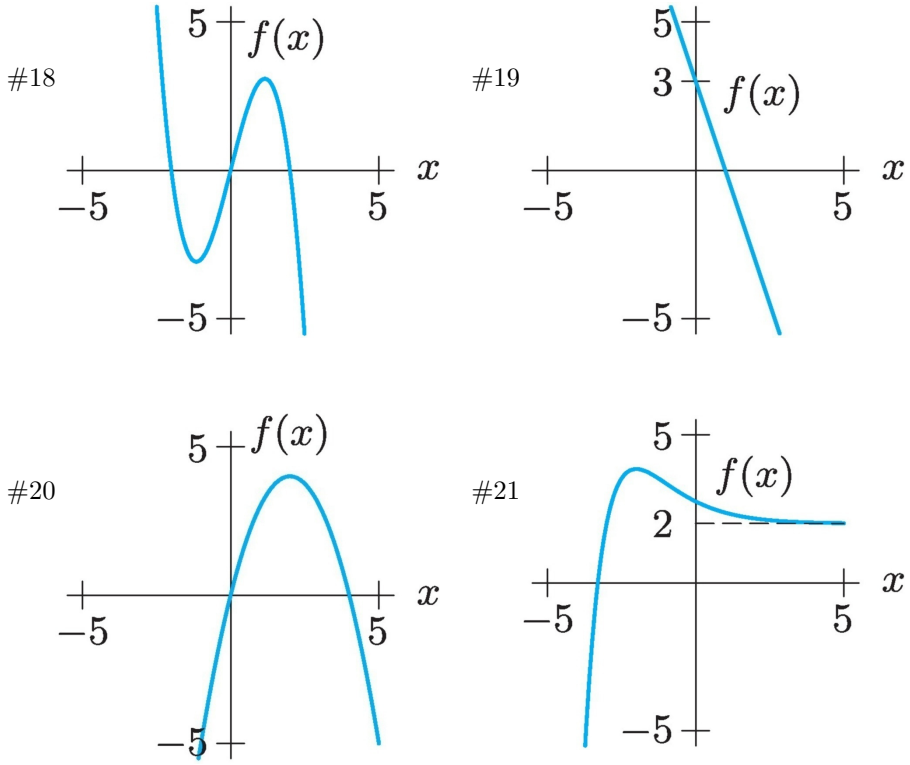
so

$$\boxed{f'(7) = 4}$$

2.2 The derivative as a function

2.2 Example 1: problems 18–21

Match the functions in Problems 18–21 with one of the derivatives in Figure 2.25.

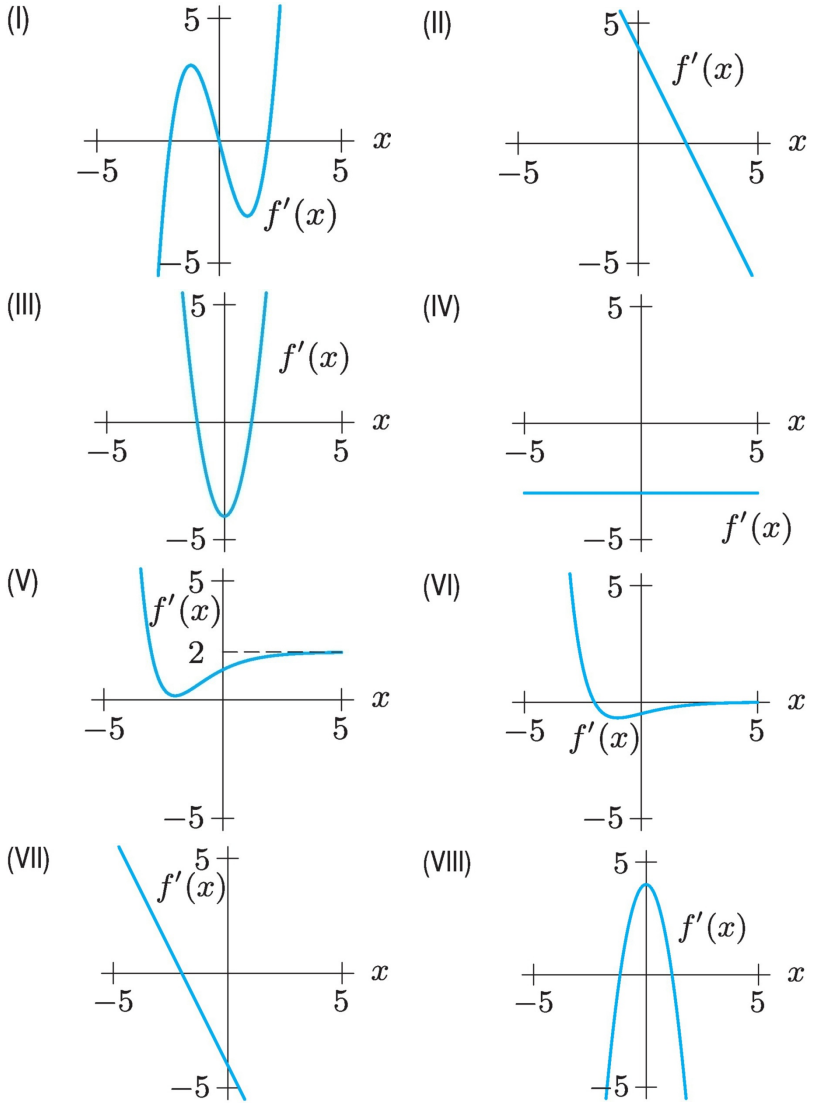


Solution: Although we have to do all four functions, we don't have to do them in order. So, we'll start with the simplest one: #19.

The function in #19 is straight line, and it always has the same slope. We may not be able to tell exactly what this slope is: maybe $m = -2$, or $m = -3$ or something like that. When we look at the graphs in Figure 2.25 we should not look at their slopes: we should look at their y-values. Which graph always has a constant y-value, of $y = -2$ or $y = -3$? Graph IV. So, #19 goes with graph IV.

Now let's look at #20. Around $x = 0$ this has a positive slope: maybe $m = 1$ or $m = 2$. Around $x = 2$ this graph is horizontal, so $m = 0$. Around $x = 4$ his has a negative slope: maybe $m = -1$ or $m = -2$. When we look at the graphs in Figure 2.25 we should not look at their slopes: we should look at their y-values. Look at $x = 0$, and $x = 2$ and $x = 4$, which graph has $y = -1$ or $y = -2$, then $y = 0$, then $y = 1$ or $y = 2$? Graph II. So, #20 goes with graph II.

Figure 2.25


(I)
5
−5
5
−5
x
f′(x)
(II)
5
−5
5
−5
x
f′(x)
(III)
5
−5
5
−5
x
f′(x)
(IV)
5
−5
5
−5
x
f′(x)
(V)
5
−5
5
−5
x
f′(x)
2
(VI)
5
−5
5
−5
x
f′(x)
(VII)
5
−5
5
−5
x
f′(x)
(VIII)
5
−5
5
−5
x
f′(x)

Now let,s look at #18. This one is more complicated, but paradoxically, this means we don't need to be as precise. There are two spots where the slope is 0: at $x = -1$ and $x = 1$. To the far left the slope is negative; between $x = -1$ and $x = 1$ the slope is positive, and to the far right the slope is negative again. When we look at the graphs in Figure 2.25 we should not look at their slopes: we should look at their y-values. Reading from left to right, we should look for y-values that are negative, 0, positive, 0, negative, or, to use shorthand:

$$-, 0, +, 0, -$$

Which graph has y-values that follow this pattern? Graph VIII.

Finally, let's look at #21. See if you can figure out why this function goes with graph VI.

2.3 Variations on the derivative

2.3 Example 1: Problems 2 and 4

Write the Leibniz notation for the derivative of the given function and include units.

#2. The cost, C, of a steak, in dollars, is a function of the weight, W, of the steak, in pounds.

#4. An employee's pay, P, in dollars, for a week is a function of the number of hours worked, H.

Solution:

#2. Since C is a function of W, we write $C = f(W)$. The Leibniz notation is $\dfrac{dC}{dW}$ and the units are dollars per pound.

#4. Since P is a function of H we write $P = f(H)$. The Leibniz notation is $\dfrac{dP}{dH}$ and the units are dollars per hour.

2.3 Example 2: Problem 6

An economist is interested in how the price of a certain item affects its sales. At a price of $\$p$, a quantity, q, of the item is sold. If $q = f(p)$, explain the meaning of each of the following statements:

(a) $f(150) = 2000$

(b) $f'(150) = -25$

Solution:

(a) In problems like these, the best way to "explain the meaning" is to write a complete, correct, English sentence that uses the least amount of technical jargon as possible. In this case, here are some examples:

"When the price is \$150, there were 2000 items sold."
"If we price it at \$150, then we'll sell 2000 items."
"A price of \$150 leads to sales of 2000."
"We'll sell 2000 items at a price of \$150."

(b) It may help here to practice writing this in Leibniz notation:

$$\left.\frac{dq}{dp}\right|_{p=150} = -25$$

The reason this notation is useful is that it reminds us that the derivative is a rate of change, and which units are on top. Thus, we can see that the units are "items per dollar" and the rate of change is -25. As before, the best thing to do is to write this information down in a complete, correct, English sentence:

"If the price changes to \$151, we can expect to sell about 25 fewer items."

"At the price of \$150, an increase of \$1 will cause the number of items sold to decrease by \$25."

2.3 Example 3

The cost, C (in dollars), to produce ℓ liters of a chemical can be expressed as $C = f(\ell)$. Using units, explain the meaning of the following statements in terms of the chemical:

(a) $f(350) = 1750$

(b) $f'(350) = 9$

Solution:

(a) This means that it costs \$1750 to produce 350 liters of chemical.

(b) This means it will cost about \$1759 to produce 351 gallons.

2.3 Example 4: Linear Approximation

For the function $f(x) = 2\ln(x)$ first

(a) Use a table of numbers to approximate $f'(1)$, and to write the equation of the tangent line at the point $(1, 0)$.

(b) Using linear approximation and your answer to part (a), to approximate $f(1.01)$, $f(1.001)$.

Solution:

(a) Here are some of the values we look at for the difference quotient:

2nd x-value	$\dfrac{2\ln(x) - 2\ln(1)}{x-1}$
0.5000	2.7726
0.9000	2.1072
0.9900	2.0101
0.9990	2.0010
0.9999	2.0001
1.0001	1.9999
1.0010	1.9990
1.0100	1.9901
1.1000	1.9062
1.5000	1.6219

From this table, we estimate that $f'(1) = 2$. Using this, the tangent line at

$$y = m(x - x_0) + y_0$$
$$y2(x - 1)$$

(b) The basic idea here is that we can use the tangent line to approximate values on the original graph. Thus

$$f(1.01) \approx y(1.01)$$

where $f(x) = 2\ln(x)$ and $y(x) = 2(x - 1)$. Using this, we have

$$f(1.01) \approx 2(1.01 - 1) = 0.02$$
$$f(1.001) \approx 2(1.001 - 1) = 0.002$$

(As a way to double check these answers, my calculator says $\ln(1.01) = 0.009950330$ and $\ln(1.001) = 0.000999500$. Our estimates are *very* close to these answers.)

To see why this sort of approximation might be useful, given the examples we are working with, you have to use your imagination a little

bit. You need to imagine a function that we *don't* know much about. For instance, the exact amount of the total national debt as a function of, t, or the amount of fuel consumption for a truck as function of its cargo weight, w. In each case, we don't know a formula for the function. But we might know what it's current value is, and how much that value is changing. Using this, we could estimate what it's value would be tomorrow, or with a slight increase in weight.

2.3 Example 5: Problem 46

The area of Brazil's rain forest, $R = f(t)$, in million acres, is a function of the number of years, t, since 2000.

(a) Interpret $f(9) = 740$ and $f'(9) = -2.7$ in terms of Brazil's rain forests.

(b) Find and interpret the relative rate of change of $f(t)$ when $t = 9$.

Solution:

(a) $f(9) = 740$ tells us that in 2009, the area of Brazil's rain forest was 740 million acres. The formula $f'(9) = -2.7$ tells us that in 2009, the area of the rain forest is decreasing by about 2.7 million acres per year.

(b)

$$\text{Relative rate of change in 2009} = \frac{f'(9)}{f(9)} = \frac{-2.7}{740} \approx -0.00365$$

Thus in 2009, the rain forests are shrinking at a rate of about 0.365% per year.

2.3 Example 6: Problem 50(b*)

The world population in billions is predicted to be approximately $P = 7.1e^{0.011t}$ where t is in years since 2013. Estimate the relative rate of change of population in 2018 using this model and $\Delta t = 0.1$.

Solution: By definition,

$$\text{Relative change in 2018} = \frac{P'(5)}{P(5)}$$

But to use this we first need to find $P'(5)$. We don't know the shortcut formula for this yet, so we will estimate it. Recall that

$$P'(5) = \lim_{\Delta t \to 0} \frac{P(5 + \Delta t) - P(5)}{\Delta t}$$

We use this with $\Delta t = 0.1$ to estimate $P'(5)$:

$$P'(5) \approx \frac{P(5.1) - P(5)}{0.1} \approx 0.08256$$

Now we can finish this problem:

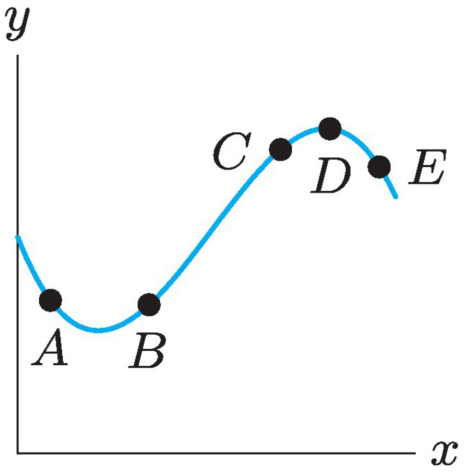
$$\text{Relative rate of change in 2018} \approx \frac{0.08256}{P(5)} \approx 0.0122858$$

Another way to put this is that the relative rate of change in 2018 is approximately 1.2286% per year.

2.4 The second derivative

2.4 Example 1: Problem 2

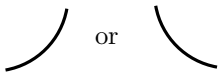
At which of the labeled points, if any, are both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ positive?



Solution: The only point where they are both positive is B.

Here's the proof. Recall that $\frac{dy}{dx}$ is positive when the slope is positive, and this means the graph is increasing. The points B and C are the only points where the slope is positive, and so we can cross off all the other points from our answer: cross off A, D, and E.

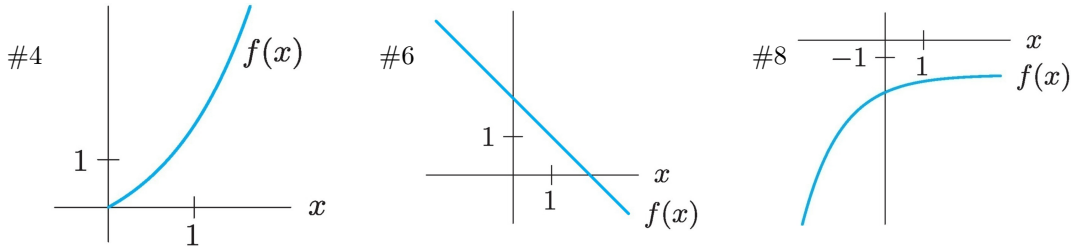
Next, recall that $\frac{d^2y}{dx^2}$ is positive where the graph is concave up, which means it must have a shape roughly like



So, the only points where $\frac{d^2y}{dx^2}$ is positive are at A and B.

2.4 Example 2: Problems 4, 6, 8

Give the signs of the first and second derivatives for the following functions. Each derivative is either positive everywhere, zero everywhere, or negative everywhere.



Solution:

#4 Since this graph is increasing, we have $f'(x)$ is positive, which is the same thing as $f'(x) > 0$. Since this graph is curving upwards, we have $f''(x)$ is positive. (Remember: Concave up is part of a cup.)

#6 Since this graph is decreasing, we have $f'(x)$ is negative, which is the same thing as $f'(x) < 0$. Since this graph is not curving at all, we have that $f''(x)$ equals 0.

#8 Since this graph is increasing, we have $f'(x)$ is positive, which is the same thing as $f'(x) > 0$. Since this graph is concave down, we have that $f''(x) < 0$. (Remember: Concave down is part of a frown.)

2.4 Example 3: Poll Everywhere

The temperature outside on a given day is given by $f(t)^\circ C$, where t is in hours since midnight. From 6 AM until noon, the first derivative was negative and the second was positive. Which of the following is correct? You may choose more than one.

This poll should be done through Poll Everywhere and then discussed online.

Directions: either go to `https://pollev.com/loberbro` or text "LOBERBRO" to the number 37607 to join. Multiple answers can be entered separated by spaces and/or commas in the same text.

(a) The temperature was below freezing but getting warmer.

(b) The temperature was below freezing and getting colder.

(c) We do not no whether the temperature was above or below freezing.

(d) The temperature was higher at noon than at 6 AM.

(e) The temperature was lower at noon than at 6 AM.

(f) The temperature was rising but at a slower rate as the morning progressed.

(g) The temperature was rising but at a faster rate as the morning progressed.

(h) The temperature was falling and at a faster rate as the morning progressed.

(i) The temperature was falling but at a slower rate as the morning progressed.

Solution: There is not an "official" solution to this, because it is meant to be a discussion.

2.4 Example 4

Let $P(t)$ represent the price of a share of stock of a corporation at time t. What does each of the following statements tell us about the signs of the first and second derivatives of $P(t)$?

(a) "The price of the stock is falling faster and faster."

(b) "The price of the stock is getting close to its peak, at which it will remain for a little while."

(c) "The price of the stock is skyrocketing."

Solution:

(a) The phrase "price ... is falling" means $P'(t) < 0$. Now you can draw three kinds of falling graphs:

Which of these do you think makes the most sense for "faster and faster." Probably the middle one. In that case, it's concave down ("concave down is part of a frown") and so $P''(t) < 0$.

(b) For the price to be getting close to its peak, we need that the graph is increasing, so $P'(t) > 0$, and starting to level out a little bit, so curving like this

That means that it's concave down, so $P''(t) < 0$.

(c) The price skyrocketing means that it is increasing, so $P'(t) > 0$. Also, it's probably not increasing more slowly as time goes on, so it's *not*  . Rather, it probably looks like this:

This means that it is concave up ("Concave up is part of a cup") and so $P''(0) > 0$.

2.5 Marginal Cost and Revenue

2.5 Example 1

It costs \$2500 to produce 1350 items and it costs \$2545 to produce 1360 items. What is the approximate marginal cost when producing 1350 items?

Solution: We don't have enough information to solve this exactly but we can approximate it:

$$\begin{aligned} MC(1350) &= \text{Derivative at } q = 1350 \\ &\approx \frac{C(1360) - C(1350)}{1360 - 1350} \\ &= \frac{2545 - 2500}{1360 - 1350} \\ &= \frac{45}{10} \\ &= 4.5 \end{aligned}$$

$$\boxed{MC \approx \$4.50 \text{ per item}}$$

2.5 Example 2: Problem 4*

Figure 2.55 shows a total cost function, $C(q)$:

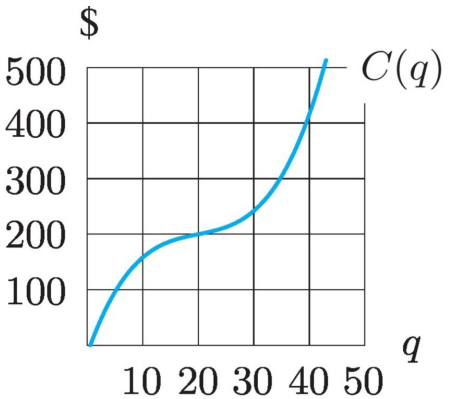


Figure 2.55

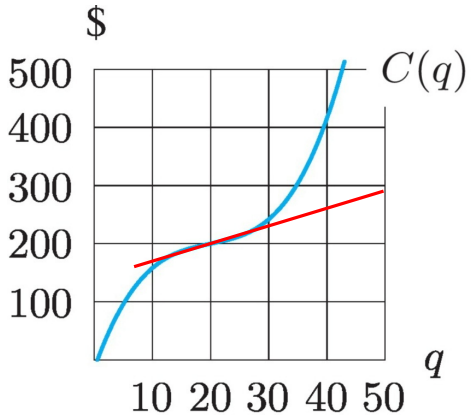
(a) Estimate the marginal cost when the production level is 20 and interpret it.

(b) Is the marginal cost greater at $q = 5$ or at $q = 30$? Explain.

(c) Is the marginal cost greater at $q = 20$ or at $q = 40$? Explain.

Solution:

(a) The best way to do this graphically is to put a ruler on the graph at $q = 20$ and make it as close to a tangent line as you can. Then draw the line and estimate some points on the line. Maybe you get something like this:



It looks like when $q = 20$, we have $C = 200$) so we have one point $(20,\ 200)$. Another point on the tangent line might be $(50,\ 300)$. This tangent line

would have slope of $m = \dfrac{300-200}{50-20} = \dfrac{100}{30} = \dfrac{10}{3}$, so marginal cost would at $q = 20$ is estimated to be about $10/3 = 3.\overline{3}$.

(b) By looking at the tangent lines when $q = 5$ and $q = 30$, both would have positive slope but at $q = 5$ the line would be steeper, thus the slope would be greater. Thus marginal cost is greater at $q = 5$ than at $q = 30$.

(c) By looking at the tangent lines when $q = 20$ and $q = 40$, both would have positive slope but at $q = 40$ the line would be steeper, thus the slope would be greater. Thus marginal cost is greater at $q = 40$ than at $q = 20$.

2.5 Example 3: Problem 8

Figure 2.57 shows part of the graph of cost and revenue for a car manufacturer. Which is greater, marginal cost or marginal revenue, at

(a) q_1?

(b) q_2?

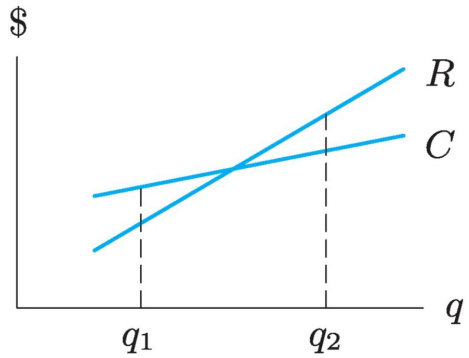


Figure 2.57

Solution: Remember *not* to look at the gap between the two lines, or at which line is higher at which point. Rather, look at the slopes of the two lines, at the points q_1 and q_2. Since the graphs are lines, the slopes don't change, and we can see that marginal revenue is higher for at both points since the slope of the revenue curve is greater than the slope of the cost curve at both points.

2.5 Example 4

To produce 2000 items, the total cost is \$4000 and the marginal cost is \$15 per item. Estimate the costs of producing:

(a) 2001 items

(b) 1999 items

(c) 2050 items

Solution:

(a) With marginal cost being \$15 when $q = 2000$, this approximates the additional cost of producing 2001 items. Since it costs \$4000 to produce 2000 items, the cost of producing 2001 items is estimated to be \$4015. We can summarize all of this in a linear equation:

$$\begin{aligned} C(2001) &\approx C(2000) + MC(2000) \times \Delta q \\ &= 4000 + 15 \times 1 \\ &= \boxed{\$4015} \end{aligned}$$

(b) With marginal cost being \$15 when $q = 2000$, this approximates the additional cost of producing 2001 items. This could also be interpreted as the amount of money saved by producing one fewer item. Since it costs \$4000 to produce 2000 items, the cost of producing 1999 items is estimated to be $\$4000 - \$15 = \$3985$:

$$\begin{aligned} C(1999) &\approx C(2000) + MC(2000) \times \Delta q \\ &= 4000 + 15 \times (-1) \\ &= \boxed{\$3985} \end{aligned}$$

(c) With marginal cost being \$15 when $q = 2000$, this approximates the additional cost of producing one more item. Thus to produce 50 more items, the additional cost is estimated to be $15(50) = 750$. Since it costs \$4000 to produce 2000 items, the cost of producing 2050 items is estimated to be $4000 + 750 = \$4750$:

$$\begin{aligned} C(2050) &\approx C(2000) + MC(2000) \times \Delta q \\ &= 4000 + 15 \times 50 \\ &= \boxed{\$4750} \end{aligned}$$

2.5 Example 5: Problem 12

Cost and revenue functions for a charter bus company are shown in Figure 2.58. Should the company add a 50th bus? How about a 90th? Explain your answers using marginal revenue and marginal cost.

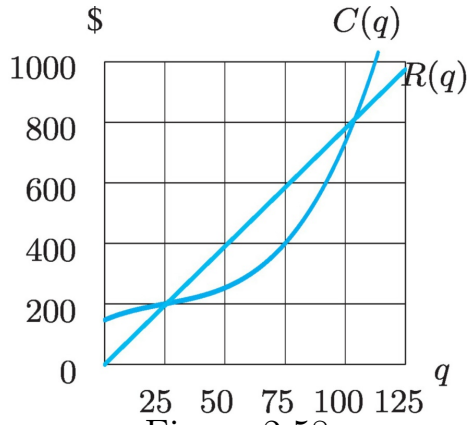
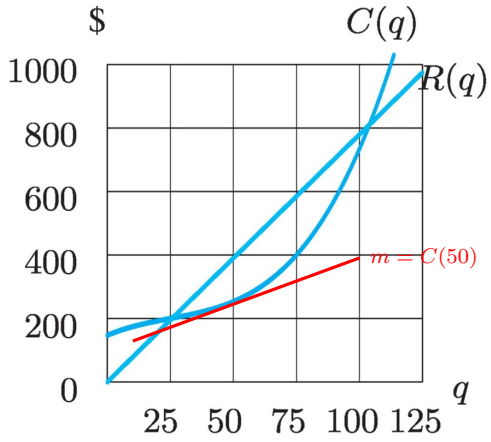


Figure 2.58

Solution: We need to look at the difference between the marginal costs and marginal revenues. At $q = 50$, the marginal revenue (slope at $R(50)$) is greater than the marginal cost (slope at $C(50)$)



So the additional revenue of adding the 50th bus will be greater than the additional cost. So yes, it should add a 50th bus.

At $q = 90$, the marginal revenue (slope at $R(90)$) is less than the marginal cost (slope at $C(90)$, so no, the 90th bus should not be added.

Chapter 3

Rules for Derivatives

3.1 Shortcuts for powers of x, constants, sums, and differences

3.1 Example 1
Let $f(x) = 3x^2 - 5x + 8$. Find $f'(x)$.

Solution: With practice, you can probably just write this down in one step:

$$f'(x) = 6x - 5$$

But when you are still learning these steps, it might help to break it down more:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(3x^2 - 5x + 8) && \text{“}f'(x)\text{” means “take the derivative”} \\
&= \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(8) && \text{apply the derivative across + and − signs} \\
&= 3 \cdot 2x^1 - 5 \cdot x^0 + 0 && \text{“constant multiple”, “constant rule” and “power rule”} \\
&= 6x - 5 && \text{just cleaning up}
\end{aligned}$$

3.1 Example 2

(a) For $f(x) = 6\sqrt{x}$ find $f'(x)$.

(b) For $C(q) = q^{13} - \dfrac{5}{q^3} + 7$, find the marginal cost.

Solution:

(a) Before taking the derivative of $\sqrt{x}$ we should rewrite it in exponential form, i.e. as a power of x:

$$6\sqrt{x} = 6x^{1/2}$$

It's very important that you become comfortable with fractional exponents: fractional exponents mean you have a root.

Now we can take the derivative:

$$\begin{aligned} f'(x) &= \frac{d}{dx} 6\sqrt{x} \\ &= \frac{d}{dx} 6x^{1/2} \\ &= 6 \cdot \frac{1}{2} x^{\frac{1}{2}-1} \\ &= 3x^{-1/2} \quad \text{or} \quad \frac{3}{\sqrt{x}} \end{aligned}$$

(b) Before taking the derivative of $\dfrac{5}{q^3}$ we should rewrite it in an exponential form, i.e. as a constant times a power of q:

$$\frac{5}{q^3} = 5q^{-3}$$

It's very important that you become comfortable with negative exponents: negative exponents mean you have "one over ...".

Now we can take the derivative:

$$\begin{aligned} C'(q) &= \frac{d}{dq}(q^{13} - \frac{5}{q^3} + 7) \\ &= \frac{d}{dq}(q^{13} - 5q^{-3} + 7) \\ &= 13q^{13-1} - 5(-3)q^{-3-1} + 0 \\ &= 13q^{12} + 15q^{-4} \quad \text{or} \quad 13q^{12} + \frac{15}{q^4} \end{aligned}$$

3.1 Example 3

(a) For $y = 2.5q^2 - 0.75q + 9.23$, find y''.

(b) For $C(q) = q(q^2 + q^{-2})$, find $C''(q)$.

Solution:

(a)

$$\begin{aligned} y' &= \frac{d}{dq}(2.5q^2 - 0.75q + 9.23) \\ &= 2.5(2)q^{2-1} - 0.75(1)q^0 + 0 \quad \text{clean up before taking next derivative} \\ &= 5q - 0.75 \\ y'' &= \frac{d}{dq}(5q - 0.75) \\ &= 5 \end{aligned}$$

(b) In this case we should "clean up" *before* you take even the first derivative:

$$C(q) = q(q^2 + q^{-2}) = q^3 + q^{-1}$$

Now we take derivatives:

$$\begin{aligned} C'(q) &= \frac{d}{dq}(q^3 + q^{-1}) \\ &= 3q^2 + (-1)q^{-1-1} \\ &= 3q^2 - q^{-2} \\ C''(q) &= 3(2)q^1 - (-2)q^{-2-1} \\ &= 6q + 2q^{-3} \end{aligned}$$

3.1 Example 4

Let $f(x) = 3x^2 - 4x + 1$.

(a) Find the equation of the tangent line to f at $(1, 0)$

(b) Find when f has a horizontal tangent line.

Solution:

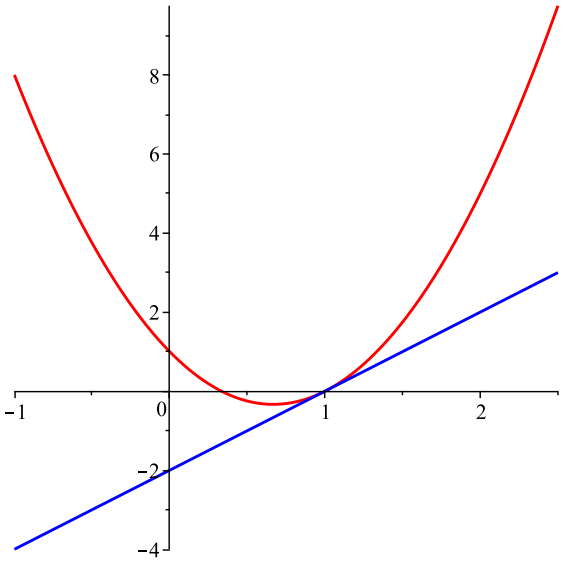
(a) We will fill in the following equation

$$\begin{aligned} y &= m(x - x_0) + y_0 \\ x_0 &= 1 \\ y_0 &= 0 \\ m &= f'(x_0) = f'(1) \end{aligned}$$

We start by taking the derivative, and then plug this into the equations above:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^2 - 4x + 1) \\ &= 6x - 4 \\ m &= f'(1) \\ &= 6(1) - 4 = 2 \\ y &= 2(x - 1) + 0 \\ &= 2x - 2 \end{aligned}$$

Just to double check, you can look at the graphs to see we got it right:



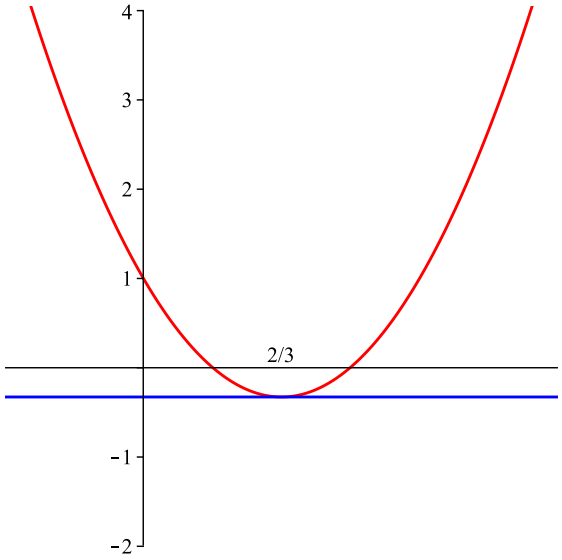
(b) First we "rewrite" the question, at least in our heads:

$$\begin{aligned} f \text{ has a horizontal tangent line} &= \text{ means the slope is } 0 \\ &= \text{ means } f'(x) = 0 \end{aligned}$$

Now we set this up as an equation and solve it:

$$\begin{aligned} f'(x) &= 0 \\ 6x - 4 &= 0 \\ 6x &= 4 \\ x &= 4/6 = 2/3 \end{aligned}$$

Just to double check, you can look at the graphs to see we got it right:



3.2 Derivatives of exponentials and logarithms

3.2 Example 1

Let $f(x) = 3x^3 + 2e^x$

(a) Find $f'(x)$.

(b) Find the equation of the tangent line at $x = 0$.

(c) Compare the graph of $f(x)$ and the graph of the tangent line.

Solution:

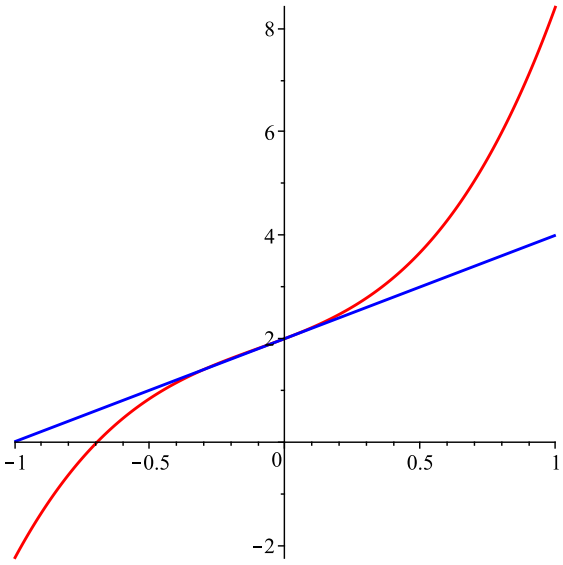
(a) Using the constant multiple rule, sum rule, and exponential rules:

$$f'(x) = 9x^2 + 2e^x$$

(b)

$$\begin{aligned} y &= m(x - x_0) + y_0 \\ x_0 &= 0 \\ y_0 &= f(0) = 2 \\ m &= f'(0) = 2 \\ y &= 2x + 2 \end{aligned}$$

(c) We show $f(x)$ and $y = 2x + 2$ below.



3.2 Example 2

The human population of the entire world can be modeled by $P = 6.8(1.011)^t$ where P is in billions, and t is the year with $t = 0$ corresponding to 2010 (source Wikipedia).

Find the estimated rate of growth in 2020, and interpret your answer, with units.

Solution: Recall that

$$\text{rate of growth} = \text{derivative}.$$

Also, the year 2020 means that $t = 10$. Thus, we need to calculate $P'(10)$, or in Leibniz notation, $\left.\frac{dP}{dt}\right|_{t=10}$. Always find the derivative as a formula first, and then plug in the number. We use the Constant Multiple Rule and the General Exponential Rule

$$\begin{aligned} P'(t) &= \frac{d}{dt}(6.8(1.011)t) \\ &= 6.8\left((1.011)t\right)' \\ &= 6.8\ln(1.011)(1.011)^t \end{aligned}$$

and plug in $t = 10$:

$$P'(10) = 6.8\ln(1.011)(1.011)^{10} \approx 0.083 \text{ billion people/year}.$$

Here is a simple interpretation:

> "In 2020 the population will increase by 0.083 billion people each year."

Note: A simpler, and better, statement would convert 0.083 billion to 83 million. I didn't do that above just because I didn't want to introduce an extra calculation or step into the solution.

3.2 Example 3

Find the marginal revenue function if $R(q) = 4q^2 + 7\ln(q)$.

Solution: Basically "marginal revenue" means we should take the derivative of $R(q)$. Really this means we should combine the following basic rules for derivatives:

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$
$$\frac{d}{dx}c \cdot f(x) = c \cdot \frac{d}{dx}f(x)$$

Now we apply these rules, as well as the power rule from earlier:

$$\begin{aligned} MR(q) &= R'(q) \\ &= \frac{d}{dq}\big(4q^2 + 7\ln(q)\big) \\ &= 4 \cdot 2q^1 + 7\frac{d}{dq}\ln(q) \\ &= 8q + 7 \cdot \frac{1}{q} \\ &= 8q + \frac{7}{q} \end{aligned}$$

3.3 The Chain Rule

3.3 Example 1

Suppose we are given $A(t) = 1000e^{0.149t}$. Find $A'(1)$.

Solution: The main idea here is to apply the chain rule to $e^{0.149t}$. If it was just e^t we would use the basic rule, $\frac{d}{dt}e^t = e^t$, i.e. do nothing. Since we have $0.149t$ "inside" of the exponent, we should use the chain rule. We start with the basic rule (do nothing in this case), we don't change the inside, and then we multiply by the derivative of the inside. I'll write this in two ways: (1) using an extra variable, z, like the book does, to label the inside function, (2) using colors and words to keep track of what's inside and what's outside. *You* don't need to write it both ways, choose what works best for you.

$$\begin{aligned}\frac{d}{dt}e^z &= \frac{dy}{dz}\cdot\frac{dz}{dt} \text{ where } y = e^z, \text{ and } z = 0.149t\\ &= e^z \cdot (0.149)\\ &= e^{0.149t}(0.149)\end{aligned}$$

or $\quad \frac{d}{dt}e^{0.149t} = e^{(0.149t)} \cdot (0.149)$

deriv. of outside — don't change inside — deriv. of inside

Now we combine this with the rest of the formula for $A(t)$ and evaluate at $t = 1$

$$\begin{aligned}A'(t) &= 1000e^{0.149t}(0.149)\\ &= 149e^{0.149t}\\ A'(1) &= 149e^{0.149}\\ &\approx 172.94\end{aligned}$$

3.3 Example 2

Find the derivatives of

(a) $R = (q^3 - 5q + 7)^5$

(b) $h(x) = \dfrac{17}{\sqrt{3 + 5x^2}}$

Solution: For Chain Rule problems, we want to think of what is the "outside function" and what is the "inside function", and we need each of these to be simple enough that we know how to find its derivatives. Sometimes it is best to think of the "outside function" as the last calculation you do.

For (a), we see that the last calculation that would be done would be to take the fifth-power what is in the parentheses. Thus we'd have $z = q^3 - 5q + 7$ and $R = z^5$. The Chain Rule gives us

$$\begin{aligned} \frac{dR}{dq} &= \frac{dR}{dz} \cdot \frac{dz}{dq}, \text{ where } R = z^5, z = q^3 - 5q + 7 \\ &= 5z^4 \cdot (3q^2 - 5) \\ &= 5(q^3 - 5q + 7)^4 (3q^2 - 5) \end{aligned}$$

or $$\frac{d}{dq}(q^3 - 5q + 7)^5 = 5(q^3 - 5q + 7)^4 \cdot (3q^2 - 5)$$

deriv. of outside — don't change inside — deriv. of inside

For (b), we need to rewrite the function, changing the radical in the denominator to a negative exponent: $h(x) = 17(3 + 5x^2)^{-1/2}$. Then we use the Power and Chain Rule Combined:

$$\frac{d}{dx} 17(3 + 5x^2)^{-1/2} = 17\left(-\tfrac{1}{2}\right)(3 + 5x^2)^{-3/2} \cdot (10x)$$

deriv. of outside — don't change inside — deriv. of inside

We can clean this up, but please be clear in your head: we are done with the derivative at this point. The rest is algebra which *might* be useful, but you also

might sometimes not want to do it, and in any case it's not the new part here.

$$\begin{aligned} 17\left(-\frac{1}{2}\right)(3+5x^2)^{-3/2}(10x) &= -\frac{17(10)}{2}(3+5x^2)^{-3/2}x \\ &= \frac{85x}{(3+5x^2)^{3/2}} \end{aligned}$$

3.3 Example 3

Find the derivatives of

(a) $y = 5e^{6x} + e^{-x}$

(b) $y = e^{3x^2-7x+11}$

Solution: With exponential functions it's a little tricky to spot what "inside function" means. Basically, we should just learn with experience that for these, "inside" means what's on top. You can make this a little easier to see if you realize that the exponents are basically inside implied parentheses:

$$e^x = e^{(x)}, \quad e^{6x} = e^{(6x)}, \quad e^{-x} = e^{(-x)}, \quad e^{3x^2-7x+11} = e^{(3x^2-7x+11)}, \text{ etc.}$$

Now "inside" really means inside, it's inside the parentheses. If you combine this with the Chain Rule you get the following pattern

$$\begin{aligned} &y = e^z,\ z = \text{stuff} && \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \\ &y = e^{(\text{stuff})} \implies y' = e^{(\text{stuff})}(\text{stuff})' \end{aligned}$$

We'll use this in both parts below.

For (a), we use the Chain Rule on each of the terms, using the pattern above. Thus we get

$$\begin{aligned} y' &= (5e^{6x} + e^{-x})' \\ &= 5e^{(6x)}(6x)' + e^{(-x)}(-x)' \\ &= 5e^{6x}(6) + e^{-x}(-1) \\ &= 30e^{6x} - e^{-x} \end{aligned}$$

For (b), we use the pattern for Exponential Functions and Chain Rule Combined:

$$\begin{aligned} y' &= e^{(\text{stuff})}(\text{stuff})' \\ y' &= e^{(3x^2-7x+11)}(3x^2 - 7x + 11)' \\ &= e^{(3x^2-7x+11)}(6x - 7) \\ &= (6x - 7)e^{3x^2-7x+11} \end{aligned}$$

Note: it's important that you include the parentheses around $6x - 7$: Without them it's not right, and you will lose points.

3.3 Example 4

Find the derivative of $f(x) = \ln(x^2 + 5)$.

Solution: Here we have: outside $= \ln(\quad)$ and inside $= x^2 + 5$.

$$\begin{aligned} \frac{df}{dx} &= \frac{df}{dz} \cdot \frac{dz}{dx} \text{ where } z = x^2 + 5 \\ &= \frac{1}{z} \cdot (2x) \\ &= \frac{2x}{x^2 + 5} \end{aligned}$$

or $\dfrac{d}{dx}\ln(x^2 + 5) = \dfrac{1}{x^2 + 5} \cdot (2x)$

deriv. of outside

don't change inside

deriv. of inside

This gives us the pattern for natural logarithms with the Chain Rule:

$$y = \ln(\text{stuff}) \implies y' = \frac{1}{(\text{stuff})}(\text{stuff})' = \frac{(\text{stuff})'}{\text{stuff}}$$

3.4 Product and Quotient Rules

3.4 Example 1

Use the Product Rule to differentiate $y = x^2 e^{5x-3}$.

Solution: When you are first learning the product rule, you may need to take a couple of extra steps to keep the notation straight: (1) Write the product rule formula on your paper, right in your solution. If you write this every time, I promise that you'll be able to remember it better, and you'll be able to use it better. (2) Label the parts of the function with f and g, then find f' and g' and then put the parts together where they go in the product rule.

$$\begin{aligned}
(fg)' &= f' \cdot g + f \cdot g' \\
f &= x^2 \\
g &= e^{5x-3} \\
f' &= 2x \\
g' &= e^{(\text{stuff})} \cdot (\text{stuff})' \\
&= e^{5x-3}(5) \\
y' &= f' \cdot g + f \cdot g' \\
&= 2x \cdot e^{5x-3} + x^2 \cdot e^{5x-3}(5)
\end{aligned}$$

It's not really necessary to do anything else to this answer: it's not going to simplify much, and we weren't asked to factor it or anything. It is traditional to put that last "5" in front of the x^2, but it's not mandatory.

Let me show you how you might solve this problem if you've been doing product rules for a while. In that case, you might not label f and g and f' and g'. You might just identify them in your head, and say the following words to yourself as you go: "derivative of the 1st, times the second, plus the 1st times the derivative of the second."

$$\underbrace{2x}_{\text{deriv. of 1st}} \cdot \underbrace{e^{5x-3}}_{\text{times 2nd}} + \underbrace{x^2}_{\text{1st}} \cdot \underbrace{e^{5x-3}(5)}_{\text{2nd deriv of times}}$$

That may seem like a lot to keep track of, or do in your head, but if you just write and say one part at a time, it's not so bad. In any case, do not feel like you should try to do more in your head; feel free to label everything and arrange all the parts like we did in our first solution above. Just know, that if you want to, you can start to label less as you go.

3.4 Example 2

Find the derivatives of the following:

(a) $y = x^3(2x-7)^4$

(b) $y = 3t^4 \ln(t)$

Solution: Both of these involve the Product Rule.

For (a), the first is x^3 and the second is $(2x-7)^4$. Notice that finding the derivative of the second will involve the Chain Rule.

$$\begin{aligned} y &= \left[x^3\right]\ \left[(2x-7)^4\right] \\ y' &= \left[x^3\right]'\ (2x-7)^4 + x^3\ \left[(2x-7)^4\right]' \\ &= \left[2x^2\right](2x-7)^4 + x^3\left[4(2x-7)^3(2)\right] \\ &= 2x^2(2x-7)^4 + 8x^3(2x-7)^3 \end{aligned}$$

Again, you probably shouldn't try to "simplify" any farther here. We've taken the derivative, and we're done with that; unless there's a reason to factor it, just stop.

For (b), in addition to the Power Rule and Product Rule, we need to remember that $\dfrac{d}{dt}\ln(t) = \dfrac{1}{t}$.

$$\begin{aligned} y &= \left[3t^4\right]\left[\ln(t)\right] \\ y' &= \left[3t^4\right]'\ln(t) + 3t^4\left[\ln(t)\right]' \\ &= 12t^3\ln(t) + 3t^4\,\frac{1}{t} \\ &= 12t^3\ln(t) + 3t^3 \end{aligned}$$

3.4 Example 3

Use the Quotient Rule to differentiate $y = \dfrac{4t+5}{2-3t^2}$.

Solution: When you are first learning the Quotient Rule, you may need to take a couple of extra steps to keep the notation straight: (1) Write the Quotient Rule formula on your paper, right in your solution. If you write this every time, I promise that you'll be able to remember it better, and you'll be able to use it better. (2) Label the parts of the function with f and g, then find f' and g' and then put the parts together where they go in the Quotient Rule.

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{(g)^2}$$
$$f = 4t+5$$
$$g = 2-3t^2$$
$$f' = 4$$
$$g' = -6t$$
$$\frac{f' \cdot g - f \cdot g'}{g^2} = \frac{(4)(2-3t^2) - (4t+5)(-6t)}{(2-3t^2)^2}$$

As before, just leave it alone and don't try to simplify, unless you've been asked to, or if there's a reason to. In other words, **simplify at your own risk**. (In case you want to practice, see if you can get $\dfrac{12t^2+30t+8}{(2-3t^2)^2}$.)

We can also learn to take the quotient rule without labelling everything, just like we talked about with the Product Rule. Here's how it looks. Say the following words to yourself as you write things down: "derivative of the top, times the bottom, minus the top, times the derivative of the bottom, all over the bottom squared."

$$\frac{\overbrace{(4)}^{\text{deriv top}}\overbrace{(2-3t^2)}^{\text{times bottom}} - \overbrace{(4t+5)}^{\text{top}}\overbrace{(-6t)}^{\text{times deriv bottom}}}{\underbrace{(2-3t^2)^2}_{\text{bottom squared}}}$$

There's a rhyming mnemonic for this as well: "low d-hi minus hi d-low, over the bottom squared, and away we go!". Here, "low d-hi" means derivative of the top (d-hi) times the bottom, etc.

3.4 Example 4

Let $f(x) = \dfrac{e^x}{2x + e^x}$.

(a) Find $f'(x)$.

(b) Find the equation of the tangent line at $x = 0$.

Solution: For part (a), we need to find $f'(x)$ (using the Quotient Rule):

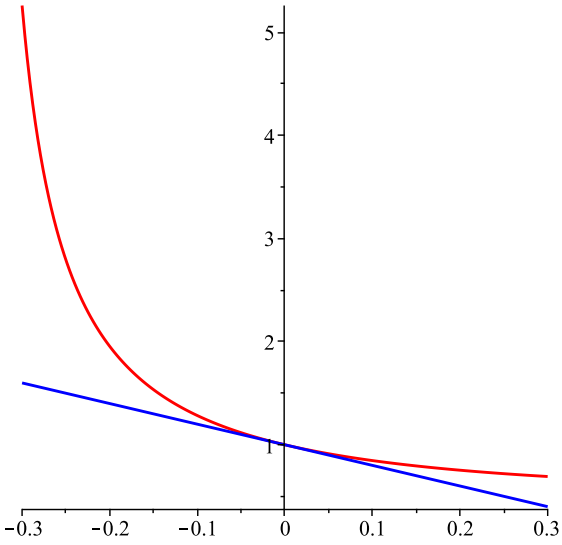
$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{(g)^2}$$
$$f'(x) = \frac{e^x(2x + e^x) - e^x(2 + e^x)}{(2x + e^x)^2}$$

Note that we're using f differently in the second line than in the first one. That's one reason we have to understand the Product Rule and Quotient Rule as more than just moving letters around: we need to understand that in a certain place it's the derivative of the top, not just the letter f'.

For part (b), we'll fill in (as always) the point-slope equation of a line:

$$\begin{aligned}
y &= m(x - x_0) + y_0 \\
x_0 &= 0 \quad \text{given above} \\
y_0 &= f(0) \\
&= \frac{e^0}{0 + e^0} \\
&= 1 \\
m &= f'(0) \\
&= \frac{e^0(0 + e^0) - e^0(2 + e^0)}{(0 + e^0)^2} \\
&= \frac{1(1) - 1(3)}{1^2} \\
&= -2 \\
y &= -2(x - 0) + 1 \\
&= -2x + 1
\end{aligned}$$

They didn't ask us for it, but just for kicks, here's a graph of $f(x)$ and the tangent line, just to show that we got it right:


5
4
3
2
1
−0.3
−0.2
−0.1
0
0.1
0.2
0.3

Chapter 4

Using the Derivative

4.1 Local Max and Mins

4.1 Intro: Definitions

Suppose c is in the domain of f:

- f has a *local maximum* at $x = c$ if $f(c) \geq f(x)$ for x near c.
- f has a *local minimum* at $x = c$ if $f(c) \leq f(x)$ for x near c.

4.1 Intro: Definitions

The point $(c, f(c))$ is a *critical point* of a function f if either

- $f'(c) = 0$ or
- $f'(c)$ is undefined.

The x-value, c, is a **critical number** of f. The y-value, $f(c)$, is a **critical value** of f.

4.1 Example 1, #5*, Discussion Part A

(a) Sketch the graph of a function with two local maxima and one local minimum.

(b) Sketch the graph of a function that has two critical points. One should be a local maximum and one should be neither a local maximum nor local minimum.

Solution: There is no "solution" for this example: we will discuss it as a group.

4.1 First Derivative Test

Suppose c is a critical point of a continuous function f. When moving from left to right:

- If $f'(x)$ changes from positive to negative at c, then f has a *local maximum* at c.
- If $f'(x)$ changes from negative to positive at c, then f has a *local minimum* at c.
- If $f'(x)$ does not change sign from at c, then f does not have a local extremum at c.

4.1 Second Derivative Test

Suppose c is a critical number for f and $f'(c) = 0$.

- If $f''(c) < 0$, then f has a *local minimum* at c.
- If $f''(c) > 0$, then f has a *local maximum* at c.
- If $f''(c) = 0$, then the Second Derivative Test tells us nothing.

4.1 Example 2

Find all local extrema of the function below, using the Second Derivative Test:

$$f(x) = \frac{2}{3}x^3 - 4x^2 - 42x.$$

Solution: Recall the main steps we need for the Second Derivative Test:

(a) Find $f'(x)$

(b) Find critical numbers (when $f'(x) = 0$ or $f'(x)$ DNE)

(c) Find $f''(x)$.

(d) Plug the critical numbers into $f''(x)$ to determine whether the critical number(s) are local extrema.

We start by finding $f'(x)$ and setting it equal to 0:

$$\begin{aligned} f'(x) &= 2x^2 - 8x - 42 \\ 2x^2 - 8x - 42 &= 0 \\ 2(x^2 - 4x - 21) &= 0 \\ 2(x-7)(x+3) &= 0 \\ x &= 7, -3 \end{aligned}$$

Now we find $f''(x)$ and plug in these critical numbers:

$$\begin{aligned} f''(x) &= 4x - 8 \\ f''(7) &= 4(7) - 8 > 0 \implies \text{local min} \\ f''(-3) &= 4(-3) - 8 < 0 \implies \text{local max} \end{aligned}$$

Finally, we find the critical *points*, i.e. the y-values that go along with each critical number. We do this by plugging the critical number back into f:

$$\begin{aligned} f(-3) &= \frac{2}{3}(-3)^3 - 4(-3)^2 - 42(-3) = 72 \\ f(7) &= \cdots = \frac{-784}{3} \approx -261.333 \end{aligned}$$

Thus the function has two critical points: $(-3, 72)$, which is a local maximum, and $(7, -784/3)$ which is a local minimum.

4.1 Example 3

Find and classify all the critical points of the function

$$f(x) = 2x^5(2x-1)^4 + 7.$$

Solution: We need to find the critical numbers by first finding $f'(x)$. This one involves the Product Rule, and the Chain Rule when we take the derivative of the second part (the $(2x-1)^4$).

$$f'(x) = 10x^4(2x-1)^4 + 2x^5\left(4(2x-1)^3(2)\right)$$

Now we set it equal to 0 and solve:

$$\begin{aligned} f'(x) &= 0 \\ 10x^4(2x-1)^4 + 16x^5(2x-1)^3 &= 0 \end{aligned}$$

Note the common factors of 2, x^4, and $(2x-1)^3$:

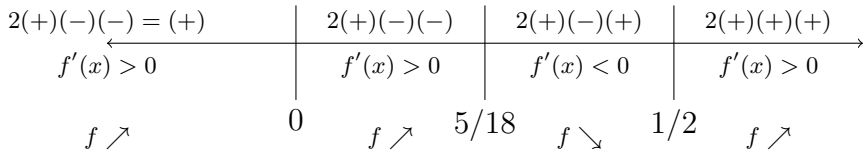
$$\begin{aligned} 2x^4(2x-1)^3\Big(5(2x-1)+8x\Big) &= 0 \\ 2x^4(2x-1)^3\left(10x-5+8x\right) &= 0 \\ 2x^4(2x-1)^3(18x-5) &= 0 \\ 2x^4 = 0 \text{ or } (2x-1)^3 = 0 \text{ or } (18x-5) &= 0 \\ x = 0, 1/2, 5/18 & \end{aligned}$$

It would take a lot of work (multiple Product Rules) to find $f''(x)$ to use the Second Derivative Test. Thus the First Derivative Test might be better in this case. So we want to find when $f'(x)$ is negative and positive, and when the derivative changes sign to find the local extrema (if any).

So we look at $f'(x) = 2x^4(2x-1)^3(18x-5)$ and we see that the only time $f'(x)$ changes sign is when it equals zero, which are the critical numbers: $x = 0$, $5/18$, and $1/2$. Note that $5/18 \approx 0.277778$.

To organize my work, I draw a number line with the critical numbers marked. Then I calculate $f'(x)$ on each side of each critical number. Note that I don't calculate the actual value, just whether the derivative is positive or negative. This makes things *much* faster. For example, we are looking at $f'(x) = 2x^4(2x-1)^3(18x-5)$ and wondering whether $f'(x)$ is positive or negative to the left of 0 (for $x < 0$). So I choose one value, say $x = -1$, and plug it into $f'(x)$. Since it is factored, I look at whether each factor is positive or negative, keeping in mind whether the powers are odd or even. So for $f'(-1)$ I get: $2(+)(-)(-) = (+)$, and mark this on the number line. We could have chosen a different value to plug in, like $x = -2$, or $x = -1/2$, or $x = -10$ and the net result would have been the same. We need to pick three more values in the other parts of the number line. Here are an easy three to use: $x = 4/18$, $x = 6/18$, and $x = 1$ (I used something over 18 to make the factor $(18x-5)$ easy to figure out). Here's the finished result

$$f'(x) = 2x^4(2x-1)^3(18x-5)$$



From the above information and the First Derivative Test, we see that there is a local maximum at $x = 5/18$, and a local minimum at $x = 1/2$, and the critical point at $x = 0$ is not a local extremum. But they asked for critical points, so we have to figure out the y-values of these points. Go all the way back to the original function $f(x) = 2x^5(2x-1)^4$ (not f'!) and plug in the critical numbers:

$$\begin{aligned}
f(x) &= 2x^5(2x-1)^4 + 7 \\
f(0) &= 7 \\
f(5/18) &= 2\left(\tfrac{5}{18}\right)^5\left(2\left(\tfrac{5}{18}\right)-1\right)^4 + 7 \\
&= 2\frac{5^5}{18^5}\left(-\frac{4}{9}\right)^4 + 7 \\
&= \frac{3125(256)}{944784(6561)} + 7 \\
&\approx 7.000129 \\
f(1/2) &= 2\left(\tfrac{1}{2}\right)^5\left(2(\tfrac{1}{2})-1\right)^4 + 7 \\
&= \frac{1}{16}(0) + 7 = 7
\end{aligned}$$

Finally, we can summarize our answer

$$\begin{aligned}
(0, 7) &: \text{not a local extremum} \\
(5/18, 7.000129) &: \text{local maximum} \\
(1/2, 7) &: \text{ local minimum}
\end{aligned}$$

4.2 Inflection points

4.2 Introduction

An *inflection point* for a function $f(x)$ is a point on the graph of $f(x)$ where the concavity changes.

An inflection point is where $f''(x)$ changes from positive to negative, or from negative to positive.

Solution:

4.2 Example 1

Find all critical points and inflection points of the function

$$f(x) = x^3 - 12x + 8.$$

Identify each critical point as a local max, local min, or neither.

Solution: We start by taking the derivative and setting it equal to 0:

$$\begin{aligned} f'(x) &= 3x^2 - 12 \\ 3x^2 - 12 &= 0 \\ 3(x^2 - 4) &= 0 \\ 3(x+2)(x-2) &= 0 \\ x &= -2, 2 \end{aligned}$$

Our critical numbers are -2 and 2. We'll test these with the second derivative test:

$$f''(x) = 6x$$

$f''(-2) = -12 \Rightarrow f''$ is $-$, f is C.D., graph is 

$f''(2) = 12 \Rightarrow f''$ is $+$, f is C.U., graph is

Therefore we can identify the behavior at each critical number

$$\begin{aligned} x &= -2 \text{ local max} \\ x &= 2 \text{ local min} \end{aligned}$$

Now we find the inflection points. We start by setting the second derivative equal to 0:

$$\begin{aligned} f''(x) &= 0 \\ 6x &= 0 \\ x &= 6 \end{aligned}$$

It's easy to see that $f''(x) = 6x$ is negative to the left of $x = 0$ (just plug any negative number into $6x$) and it's positive to the right of $x = 0$. Thus, $x = 0$ is the location of an inflection point.

Finally, we calculate the y-values at the critical points and the inflection point. We do this by plugging back into the original f (not f' or f''!).

$$\begin{aligned} f(x) &= x^3 - 12x + 8 \\ f(-2) &= 24 \\ f(0) &= 8 \\ f(2) &= -8 \end{aligned}$$

4.2 Example 2

Find all inflection points of each of the functions below

$$f(x) = x^9 \quad \text{and} \quad g(x) = x^6$$

Solution: We start by finding where the second derivative of each of these is 0, and then look at whether the second derivative is positive or negative on either side of these numbers.

$$\begin{aligned} f'(x) &= 9x^8 \\ f''(x) &= 72x^7 \\ 72x^7 &= 0 \\ x &= 0 \end{aligned}$$

Now we look to the left and the right of 0. For instance, is $f''(x)$ positive or negative when we look at $x = -1$?

$$\begin{aligned} f''(-1) &= - \Rightarrow f \text{ is C.D.} \\ f''(1) &= + \Rightarrow f \text{ is C.U.} \end{aligned}$$

In other words, f changes from C.D. to C.U. at $x = 0$, so $x = 0$ is the location of an inflection point.

$$\begin{aligned} g'(x) &= 6x^5 \\ g''(x) &= 30x^4 \\ 30x^4 &= 0 \\ x &= 0 \end{aligned}$$

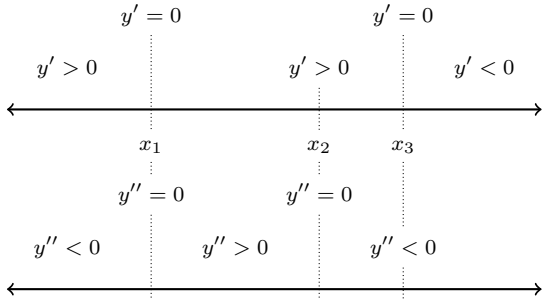
Now we look to the left and the right of 0. For instance, is $g''(x)$ positive or negative when we look at $x = -1$?

$$\begin{aligned} g''(-1) &= + \Rightarrow g \text{ is C.U.} \\ g''(1) &= + \Rightarrow g \text{ is C.U.} \end{aligned}$$

In other words, g does *not* change concavity at $x = 0$, so $x = 0$ is not the location of an inflection point.

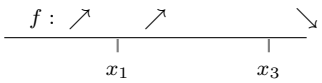
4.2 Example 3: Problem #26

Sketch a possible graph of $y = f(x)$, using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$. Assume that the function is defined and continuous for all real x.



Solution: From the information in the chart, we see that $y' = 0$ at x_1 and x_3. This means that x_1 and x_3 are critical numbers.

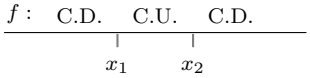
Can we say if these points are local max or mins? Yes. We have that y' is positive on both sides of x_1, and y' changes from positive to negative at x_3. We can summarize this info in a picture



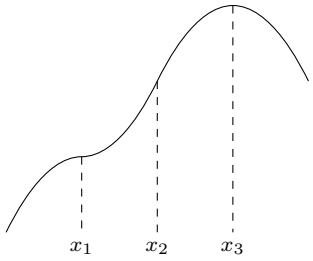
This makes it easy to see that x_1 is neither a max nor a min, but, on the other hand, x_3 is a local max.

From the information in the chart, we see that $y'' = 0$ at x_1 and x_2. This means that x_1 and x_3 are possible locations of inflection points.

Can we say if inflection points really occur here? Yes. We have that y'' changes from negative to positive at x_1, and changes back to negative again at x_2. We can summarize this info in a picture



This makes it easy to see that x_1 and x_2 are both locations of inflection points. Here's a sketch of the shape that f must have (I only claim this is the shape: the y-values could be positive or negative, the whole picture could be stretched in one direction, or compressed, etc.)


x_1
x_2
x_3

4.2 Example, Discussion Part B

Graph a function with the given properties.

(a) Has local minimum and global minimum at $x = 3$ but no local or global maximum.

(b) Has local minimum at $x = 3$, local maximum at $x = 8$, but no global maximum or minimum.

(c) Has no local or global maxima or minima.

(d) Has local and global minimum at $x = 3$, local and global maximum at $x = 8$.

Solution: Discussion opportunity.

4.3 Global max and min

4.3 Global Max/Min Test

To find the global max and global min of $f(x)$ on an interval:

(a) Find the critical numbers

(b) Compare the values for $f(x)$ at the critical numbers and at the ends of the interval.

"Values for $f(x)$" means you plug the critical numbers and ends of the interval into $f(x)$ and calculate the result.

4.3 Example 1

For the function

$$f(x) = x^5 - 2x^4, \quad -1 \le x \le 2$$

identify any global maxima and minima of f in the given interval.

Solution: We start, as always, by finding the critical numbers. In other words, we take the derivative and set it equal to 0. For $f(x) = f(x) = x^5 - 2x^4$,

$$\begin{aligned} f'(x) &= 5x^4 - 8x^3 \\ 5x^4 - 8x^3 &= 0 \\ x^3(5x - 8) &= 0 \\ x = 0, \quad x &= \frac{8}{5} \end{aligned}$$

Now it's rather easy to finish: we plug these critical numbers back into f (not f'!), and do the same thing with the end points, $x = -1$ and $x = 2$, and simply look at which produces the largest y-value and which produces the smallest y-value:

$$\begin{aligned} f(0) &= 0, \\ f(8/5) &= (8/5)^5 - 2(8/5)^4 \\ &= \frac{32768}{3125} - \frac{2(4096)}{625} \\ &= \frac{-8192}{3125} \\ &\approx -2.62144 \\ f(-1) &= -1 - 2 \\ &= -3 \\ f(2) &= 32 - 32 \\ &= 0 \end{aligned}$$

Now, simply look at the y-values: 0, -2.62, -3 and 0 again, and identify the largest and smallest. We can summarize our findings:

$$\begin{aligned} (0,0) \text{ and } (2,0): &\quad \text{Global Max} \\ (-3,-2.62): &\quad \text{Global Min} \end{aligned}$$

4.3 Example 2: Problem 32

The energy expended by a bird per day, E, depends on the time spent foraging for food per day, F hours. Foraging for a shorter time requires better territory, which then requires more energy for its defense. Find the foraging time that minimizes energy expenditure if

$$E = 0.25F + \frac{1.7}{F^2}$$

Solution: To find the minimum value of E, we do what we always do: take the derivative and set it equal to 0:

$$\begin{aligned} \frac{dE}{dF} &= 0.25 - \frac{2(1.7)}{F^3} \\ 0.25 - \frac{3.4}{F^3} &= 0 \\ F^3 &= \frac{2(1.7)}{0.25} \\ F &= \left(\frac{3.4}{0.25}\right)^{1/3} \\ &\approx 2.3870 \end{aligned}$$

Looking at the second derivative, we see it is always positive, so a foraging time of $F \approx 2.387$ hours gives a local minimum. This is the global minimum for $F > 0$ (graph it to verify).

4.4 Optimizing Cost and Revenue

4.4: Discussion Part I

Suppose you're looking at a demand curve. What does the point on the curve where $p = 0$ mean? What does the point where $q = 0$ mean?

Solution: There is no "solution" here: you'll be discussing it with your classmates and your teacher online.

4.4 Example 1: Problem 6

Let $C(q)$ be the total cost of producing a quantity q of a certain product. See Figure below (Figure 4.52 in the text).

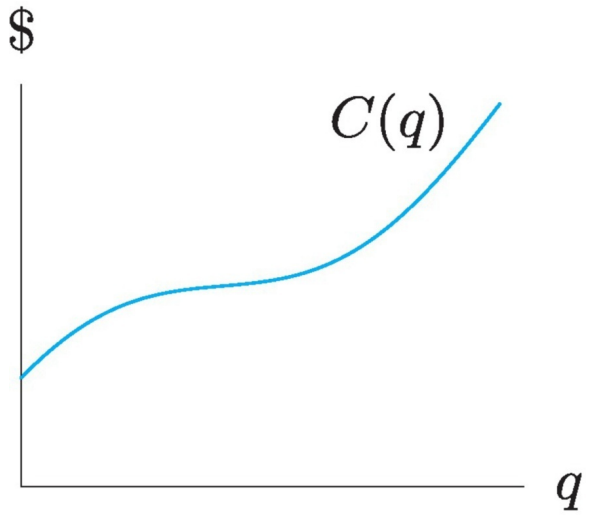
(a) What is the meaning of $C(0)$?

(b) Describe in words how the marginal cost changes as the quantity produced increases.

(c) Explain the concavity of the graph (in terms of economics).

(d) Explain the economic significance (in terms of marginal cost) of the point at which the concavity changes.

(e) Do you expect the graph of $C(q)$ to look like this for all types of products?



Solution:

(a) $C(0)$ is the amount of the fixed costs before production. This would include costs of initial investments such as the building(s), equipment, etc. needed to begin production.

(b) The marginal cost decreases slowly, and then increases as quantity produced (q) increases.

(c) Concave down implies decreasing marginal cost, while concave up implies increasing marginal cost.

(d) Since the concavity of the graph is determined by $C''(q) = MC'(q)$, when the graph is concave up, that means $C''(q) = MC'(q)$ is positive. Thus marginal cost (MC) is increasing when cost is concave up, and likewise

marginal cost is decreasing when cost is concave down. From this we get an inflection point on the graph of cost will be when marginal cost changes from increasing to decreasing, or from decreasing to increasing. Thus an inflection point of the cost function is a point where marginal cost has a local or global extremum.

(e) Because of volume pricing of supplies, etc., there are production levels in which the additional cost to produce more items (marginal cost) would decrease. But at some point, one would have to pay workers overtime, hire more workers, rent more storage space, add more delivery trucks, etc. Thus at some point the additional cost to produce more (marginal cost) will go up.

4.4 Example 2

A demand function is $p = 400 - 2q$, where q is the quantity of the good sold for price $\$p$.

(a) Find an expression for the total revenue R, in terms of q.

(b) Find the marginal revenue, MR, in terms of q. Calculate the marginal revenue when $q = 10$.

(c) Compare with the change in total revenue when production changes from $q = 10$ to $q = 11$ using the revenue function to the approximation in change in revenue using MR.

Solution: Before we start, make sure you understand the basic formula $p = 400 - 2q$. This means that we can plug in something like $q = 100$ and get the price, $p(100)$. Sometimes we do demand problems the other way around, so always stop and double check.

(a)

$$\begin{aligned} R(q) &= (\text{price})(\#\text{ sold}) \\ &= [p(q)][q] \\ &= (400 - 2q)(q) \\ &= 400q - 2q^2 \end{aligned}$$

(b)

$$\begin{aligned} MR &= R'(q) \\ &= 400 - 4q \\ MR(10) &= 400 + 4(10) \\ &= 360 \end{aligned}$$

(c)

$$\begin{aligned} \Delta R &= R(11) - R(10) \\ &= 400(11) - 2(11^2) - \Big(400(10) - 2(10^2)\Big) \\ &= 4158 - 3800 \\ &= 358 \end{aligned}$$

The point of part (c) is that 358 is really close to 360, but it's actually much easier to calculate the 360. So, rather than calculate marginal revenue by plugging in two values of q that are next to each other, and subtracting, we should use the derivative.

4.4 Example 3

The demand equation for a product is $p = 295 - 0.2q$. Write the revenue as a function of q and find the quantity that maximizes revenue. What price corresponds to this quantity? What is the total revenue at this price?

Solution: We start by finding $R(q)$.

$$\begin{aligned} R(q) &= (\text{price})(\#\text{ sold}) \\ &= [p(q)][q] \\ &= (295 - 0.2q)(q) \\ &= 259q - 0.2q^2 \end{aligned}$$

Now, as always, we take the derivative and set it equal to 0

$$\begin{aligned} R'(q) &= 295 - 0.4q \\ 295 - 0.4q &= 0 \\ q &= \frac{295}{0.4} \\ &= 737.5 \end{aligned}$$

This is a critical number. To verify that it's a maximum, note that the original function $R(q) = 295q - 0.2q^2$ is a parabola that opens downwards. Thus, the only critical point that it has is a maximum. (You can also pretty easily double check that the second derivative is negative, so again, it's a maximum.)

To find the price that corresponds to this quantity we use the original function for p:

$$\begin{aligned} p &= 295 - 0.2q \\ p(737.5) &= 295 - 0.2(737.5) \\ &= \$147.5 \end{aligned}$$

Finally, we'll get the revenue that we expect at this quantity. We can calculate it in the simplest possible way: we've got 737.5 things, and each of them sells at a price of \$147.5, for a total of

$$737.5 \times \$147.5 = \$108,781.25$$

We could also plug 737.5 directly into the formula for $R(q)$:

$$R(737.5) = 295(737.5) - 0.2(737.5)^2 = 108,781.25$$

4.4 Example 4

(a) Production of an item has fixed costs of $\$9,500$ and variable costs of \$175 per item. Express the cost, C, of producing q items.

(b) The relationship between price, p, and quantity, q, demanded is linear. Market research shows that $10,500$ items are sold when the price is \$280 and $13,000$ items are sold when the price is \$250. Express p as a function of price q.

(c) Find the profit function $P(q)$.

(d) How many items should the company produce to maximize profit? (Give your answer to the nearest integer.) What is the profit at that production level? What is the price charged at that production level?

Solution:

(a) $C(q) = 9500 + 175q$

(b) Be very carefully! Some problems, many, use q as a function of p. But we are doing it the other way this time, we want p= stuff involving q.

We solve this problem the same we solve every similar linear problem: use 2 points on the line: $(10500, 280)$ and $(13000, 250)$ so we can calculate m, and plug into the point-slope formula:

$$\begin{aligned} p - p_0 &= m(q - q_0) \\ \text{or } p &= m(q - q_0) + p_0 \end{aligned}$$

We can let $q_0 = 10500$, and $p_0 = 280$, and so all we have to do is calculate m:

$$\begin{aligned} m &= \frac{280 - 250}{10500 - 13000} = \frac{30}{-2500} = -\frac{3}{250} \\ p &= -\frac{3}{250}(q - 10500) + 280 \\ &= -\frac{3}{250}q + 126 + 280 \\ &= -\frac{3}{250}q + 406 \end{aligned}$$

(c) As always, $R(q) =$ price $\times$ quantity:

$$\begin{aligned} R(q) &= p \times q \\ R(q) &= \left(-\frac{3}{250}q + 406\right) q \end{aligned}$$

$$
\begin{aligned}
R(q) &= -\frac{3}{250}q^2 + 406q \\
P(q) &= R(q) - C(q) \\
&= -\frac{3}{250}q^2 + 406q - (9500 + 175q) \\
&= -\frac{3}{250}q^2 + 231q - 9500
\end{aligned}
$$

Finding the maximum of profit can be done in two different ways: Finding the critical point of P or by setting $MR = MC$. Let's try $MR = MC$ (recall that MC, the marginal cost, was given in the very first line of the problem):

$$
\begin{aligned}
MR &= MC \\
-\frac{6}{250}q + 406 &= 175 \\
-\frac{6}{250}q + 406 &= 175 \\
\frac{6}{250}q &= 231 \\
q &= 231(250/6) \\
&= 9625
\end{aligned}
$$

Since P is a parabola opening downwards, we know this critical point is the vertex and a maximum. To find the actual profit, we should plug $q = 9625$ into P to get $P(9625) = 1102187.5$ and to find the actual price we should use our formula in part b to get $p(9625) = \$290.5$. Now we can summarize everything we've found

$$
\begin{aligned}
\text{Maximum profit: } & \$1,102,187.5 \\
\text{price: } & \$290.5 \\
\text{quantity: } & 9625
\end{aligned}
$$

4.4: Discussion Part II

Suppose you are making something, say T-shirts, and you want to model how much revenue you'll bring in. You know the demand curve of your T-shirts, i.e. how many T-shirts you'll sell at a given price. You can write the demand curve in two ways, $q = Q(p)$, i.e. quantity sold depends on the price you set, or $p = P(q)$, i.e. the price you set should depend on how many you want to sell.

Which do you think makes more sense: use $q = Q(p)$ and write R as a function p, so $R(p) = p \times Q(p)$, or use $p = P(q)$ and write R as a function of q, so $R(q) = P(q) \times q$?

Solution: There is no "solution" here: it's an(other) opportunity to discuss something!

4.5 Average Cost

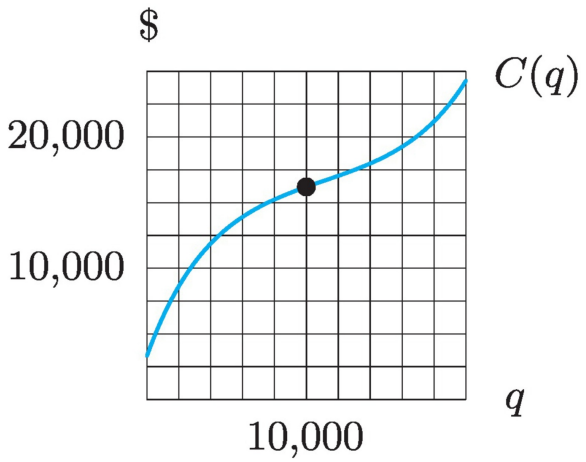
4.5 Example 1: Problem 2

Figure 4.63 shows cost with $q = 10,000$ marked.

(a) Find the average cost when the production level is 10,000 units and interpret it.

(b) Represent your answer to part (a) graphically.

(c) At approximately what production level is average cost minimized?



Solution:

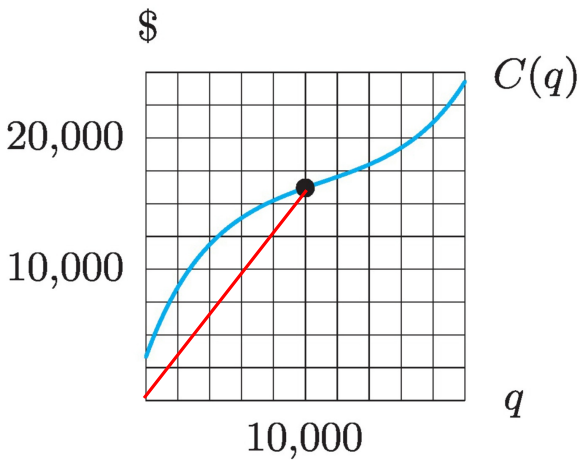
(a)

$$\begin{aligned} \text{average cost} = a(q) &= \frac{C(q)}{q} \\ &= \frac{C(10000)}{10000} \\ &\approx \frac{16000}{10000} \\ &= \$1.60 \text{ per unit} \end{aligned}$$

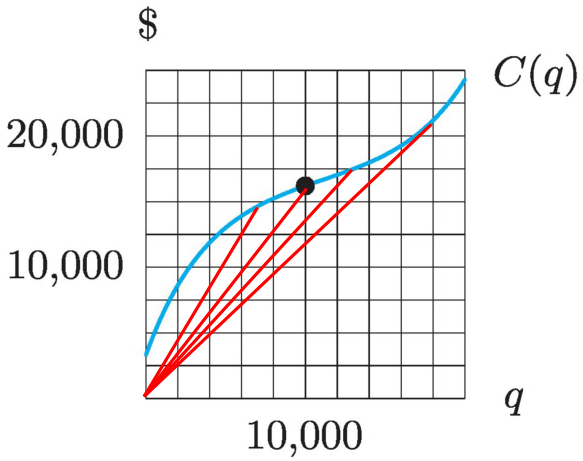
(b) Conceptually, our answer to part (a) can be viewed as the slope of a line through $(0, 0)$ and $(10000, 16000)$

$$m = \frac{16000 - 0}{10000 - 0}$$

So we can picture this slope by adding a line to the above figure:



(c) To minimize the average cost, we would look at different lines from the origin to points on the curve, and we would ask: Out of all such lines, which one has the smallest possible slope? Here I've drawn 4 of them, just to illustrate:



The last one, that goes to something like $q = 18,000$, is the least steep of all the lines we've added. In fact, it's the least steep one we *can* add. So that's about where the minimum of average cost occurs. This is really important: notice at the point $(q, C) = (18000, 20500)$ we have that the red line, which represents the average cost, is tangent to $C(q)$, the total cost. Thus, at this point, the average cost equals the marginal cost. We'll return to this in a later example.

4.5 Example 2

The cost function is $C(q) = 1000 + 20q$. Find the marginal cost to produce the 200th unit and the average cost of producing 200 units.

Solution: The marginal cost is simply the derivative

$$MC(q) = C'(q) = 20$$

In other words, marginal cost is constant, and so $MC(200) = 20$.

The average cost $C(200)/200$:

$$a(100) = \frac{C(200)}{200} = \frac{1000 + 20(200)}{200} = 25$$

In other words, the added cost of making one additional unit is about \$20, but on average the items have cost \$25 each.

4.5 Example 3: Problem 9

The average cost per item to produce q items is given by

$$a(q) = 0.01q^2 - 0.6q + 13, \text{ for } q > 0.$$

(a) What is the total cost, $C(q)$, of producing q goods?

(b) What is the minimum marginal cost? What is the practical interpretation of this result?

(c) At what production level is the average cost a minimum? What is the lowest average cost?

(d) Compute the marginal cost at $q = 30$. How does this relate to your answer to part (c)? Explain this relationship both analytically and in words.

Solution:

(a) Since $a(q) = \dfrac{C(q)}{q}$ we can turn this around and say $C(q) = a(q) \times q$. This gives

$$C(q) = 0.01q^3 - 0.6q^2 + 13q$$

(b) There are two derivatives here: MC equals the derivative of $C(q)$, but to minimize MC we should take the derivative of MC and set this equal to 0:

$$\begin{aligned} MC &= 0.03q^2 - 1.2q + 13 \\ MC' &= 0.06q - 1.2 \\ 0.06q - 1.2 &= 0 \\ q &= 1.2/0.06 \\ &= 20 \end{aligned}$$

Since MC is a parabola opening upward, we know that its only critical point is a minimum. To find out what this minimum is, we plug $q = 20$ into the formula for MC (we plug it into MC because that is the function we are finding the minimum of). We get $MC(20) = 1$ and so

$$\text{Global Minimum:} \quad MC = \$1 \text{ when } q = 20$$

On a practical level, it means that when we make 20 items, we are operating at peak efficiency in the sense that each additional item only has an added cost of \$1 per item.

(c) We will learn in this problem that there are two ways to minimize average cost: find a critical point of $a(q)$ directly, or compare $a(q)$ to $MC(q)$ (we did this earlier on graphs).

Here's how it looks to find the critical point of $a(q)$ directly:

$$\begin{aligned} a(q) &= 0.01q^2 - 0.6q + 13 \\ a'(q) &= 0.02q - 0.6 \\ 0.02q - 0.6 &= 0 \\ q &= \frac{0.6}{0.02} = 30 \end{aligned}$$

Because $a(q)$ is a parabola opening upward, this critical point is a minimum.

The average cost at this point is

$$a(30) = 0.01(30^2) - 0.6(30) + 13 = \$4 \text{ per item}$$

(d) We simply plug in $q = 30$ to our formula above for MC

$$MC(30) = 0.03(30^2) - 1.2(30) + 13 = \$4 \text{ per item}$$

This is the same as our answer to part (c). We saw in an earlier example, using graphs, that average cost should be minimized when $MC = a(q)$, and that's the same thing we found just now. Here's the "analytical" reason. What that means is that we will take the derivative of $a(q)$, and set this equal to 0, using only general formulas:

$$\begin{aligned} a(q) &= \frac{C(q)}{q} \\ a'(q) &= \frac{C'(q) \times q - C(q) \times 1}{q^2} \\ \frac{qC'(q) - C(q)}{q^2} &= 0 \\ qC'(q) - C(q) &= 0 \\ qC'(q) &= C(q) \\ C'(q) &= \frac{C(q)}{q} \\ MC &= a(q) \end{aligned}$$

This gives us the following very useful fact

average cost has a critical point when $MC(q) = a(q)$

This should make sense on a practical level as well, using only words that someone could understand without calculus. Say you want to minimize average cost (which you usually do). If at some production level, each item you make costs more than your average, then you should cut back, because you're just increasing the average cost. On the other hand,

if at some production level, each item you make costs less than your average, then you should make more, because you're lowering your average cost. When should you stay put, at the current production level? When each item costs the same as the average cost.

4.6 Elasticity of Demand

4.6 Formulas for Elasticity

4.6 Example 1

The elasticity of the demand for eggs is 0.43 and the elasticity of fresh tomatoes is 2.22. What is the effect on the quantity demanded of both eggs and tomatoes of

(a) a 10% increase in price?

(b) a 15% decrease in price?

Solution:

(a) With 10% increase in price, we expect for eggs:

$$\frac{\Delta q}{q} \approx -E\,\frac{\Delta p}{p} = -0.43(0.10) = -0.043$$

For tomatoes:

$$\frac{\Delta q}{q} \approx -E\,\frac{\Delta p}{p} = -2.22(0.10) = -0.222$$

Thus with a 10% increase in price, we can expect about 4.3% decrease in demand for eggs, and about 22.2% decrease in demand for fresh tomatoes.

(b) With With 15% decrease in price, we expect for eggs:

$$\frac{\Delta q}{q} \approx -E\,\frac{\Delta p}{p} = -0.43(-0.15) = 0.0645$$

For tomatoes:

$$\frac{\Delta q}{q} \approx -E\,\frac{\Delta p}{p} = -2.22(-0.15) = 0.333$$

Thus with a 15% decrease in price, we can expect about 6.45% increase in demand for eggs, and about 33.3% increase in demand for fresh tomatoes.

By comparing the effect change in prices are on eggs ($E < 1$) and tomatoes ($E > 1$), we can see how eggs have *inelastic demand* while fresh tomatoes have *elastic demand*.

4.6 Example 2

In Fall 2013, the undergraduate enrollment at Loyola University Maryland was 3875 and the tuition was \$41850 per year (information taken from the 2013–2014 Loyola Catalogue). According to `http://centerforcollegeaffordability.org/archives/1336` the elasticity of demand for a 4 year college is 0.10.

(a) Will a 5% increase in tuition cause total revenue to go up or go down?

(b) Can you find a way to predict this answer without repeating all the calculations?

Solution:

(a) Since the elasticity is 0.10, a 5% increase in tuition should cause a $0.1(0.05)$ decrease in attendance. Thus, the attendance is predicted to be

$$3875(1 - 0.005) \approx 3856.$$

Compare old and new revenues:

$$\begin{aligned} R &= pq \\ \text{old: } R_1 &= (41850)(3875) = \$162,168,750 \\ \text{new: } R_2 &= (41850 * 1.05)(3856) = \$169,442,280 \end{aligned}$$

So the revenue increase in tuition would cause total revenue to go up (by $\$7,273,530$)

(b) We can realize percent change in $R = pq$:

$$\begin{aligned} \%\text{ change in } R &= \frac{\Delta R}{R} \\ &= \frac{(\Delta p)q + p(\Delta q)}{p \cdot q} \\ &= \frac{(\Delta p)q}{p \cdot q} + \frac{p(\Delta q)}{p \cdot q} \\ &= \frac{\Delta p}{q} + \frac{\Delta q}{q} \\ &= \frac{\Delta p}{p} - E\frac{\Delta p}{p} \\ &= (1 - E)\frac{\Delta p}{p} \\ &= (1 - E)(\%\text{ change in } p) \end{aligned}$$

So for this problem, we have $E = 0.1$ and percent change in price is 0.05:

$$\%\text{ change in } R = (0.9)(0.05) = 0.045$$

Thus we can expect revenue to increase by about 4.5%.

From part (a), $\dfrac{7,273,530}{162,168,750} \approx 0.04485$ or about 4.49%.

4.6 Critical Points in Elasticity

4.6 Example 3

The demand function of Loyola T-shirts is $q = 1500 - 125p$.

(a) Find R when $p = \$5$.

(b) Find E when $p = \$5$.

(c) When $p = \$5$, find out if R is increasing or decreasing (i.e. will increasing p make R increase or decrease?). Do the problem in two different ways: by using the Elasticity, and by finding R as a function of p and using the derivative.

Solution:

(a) Never forget, revenue equals the the number of things you sell, times how much you sell each one for:

$$\begin{aligned} R &= pq \\ &= 5\Big(1500-125(5)\Big) \\ &= 5(875) \\ &= \$4375 \end{aligned}$$

(b)

$$\begin{aligned} E &= \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| \\ &= \left|\frac{5}{875}(-125)\right| \\ &\approx 0.714286 \end{aligned}$$

(c) There's a general rule for how E will tell us about R: If demand for a quantity is inelastic, then revenue will increase with price:

$$E < 1 \implies R \nearrow$$

where "$R \nearrow$" means that if p increases, then so does R.

We should be able to verify this directly using our formula for R: Using R and its derivative:

$$\begin{aligned} R(p) &= p(1500 - 125p) \\ &= 1500p - 125p^2 \\ \frac{dR}{dp} &= 1500 - 250p \\ \left.\frac{dR}{dp}\right|_{p=5} &= 1500 - 250(5) \\ &= 250 \end{aligned}$$

Since the rate of change of R with respect to p is positive, R will increase.

Chapter 5

Accumulated Change: the Definite Integral

5.1 Distance and Accumulated Change

5.1 Intro, Example 1

The odometer on our car is broken, but we really need an estimate of how far we're driving. The speedometer readings are shown below; use them to estimate the distance traveled over the first 30 minutes. Find a lower estimate and an upper estimate.

Time (min)	0	10	20	30
Velocity (mi/h)	17	32	35	37

Solution: To find the lower estimate, we look at each 10 minute interval, and choose the lowest velocity that we see. For example, from $t = 0$ to $t = 10$ the lowest velocity is 17 mi/h. Note that there's one last trick here: we need to convert units. A 10 minute interval is $10/60$ of an hour, so, from $t = 0$ to $t = 10$, our lower estimate for distance traveled would be $17 \cdot 10/60$. In a similar way, we get the following numbers in each interval:

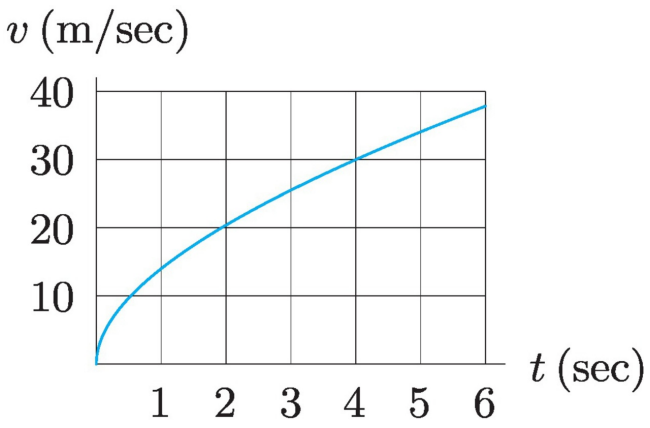
$$\begin{aligned} \text{lower estimate: } &= 17 \cdot 10/60 + 32 \cdot 10/60 + 35 \cdot 10/60 \\ &= (170 + 320 + 350)/60 \\ &= (490 + 350 = 840)/60 \\ &= 14 \end{aligned}$$

To get our upper estimate, we repeat most of the above steps, but in each 10 minute interval we use the highest velocity that we see. We get the following numbers in each interval:

$$\begin{aligned} \text{upper estimate: } &= 32 \cdot 10/60 + 35 \cdot 10/60 + 37 \cdot 10/60 \\ &= (320 + 350 + 370)/60 \\ &= (670 + 370)/60 \\ &= 1040/60 \\ &\approx 17.333 \end{aligned}$$

5.1 Example 2: Problem 7

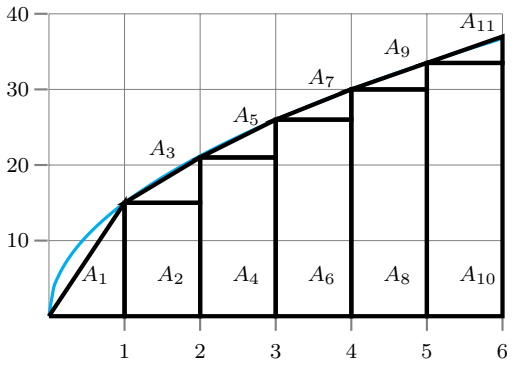
The figure below shows the velocity, v, of an object (in meters/sec). Estimate the total distance the object traveled between $t = 0$ and $t = 6$.



Solution: Let's start by focusing on the main idea here:

area below this curve = total distance traveled.

So, what we are *really* going to find is the area under the curve. We haven't been told to use a particular method to estimate this area, so we'll decide on our own how to go, and we'll split it up into rectangles and triangles, as shown



Now we simply estimate the heights of each rectangle and triangle using the numbers on the grid, and guessing when it's between two numbers. I won't spend time worring about exactly how well you read the graph, but here are the numbers I found:

$$\begin{aligned}\text{area} &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11} \\ &= \frac{1}{2}(1)(13) + 13 + \frac{7}{2} + 20 + \frac{5}{2} + 25 + \frac{5}{2} + 30 + 2 + 34 + 2 \\ &\approx 141\end{aligned}$$

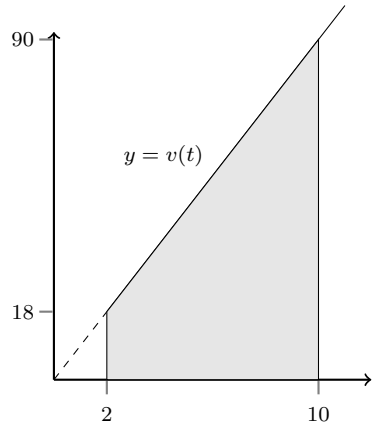
5.1 Example 3

The velocity of a vehicle on a track is given by $v(t) = 9t$ m/s. Find the exact distance traveled by this vehicle from $t = 2$ to $t = 10$ seconds.

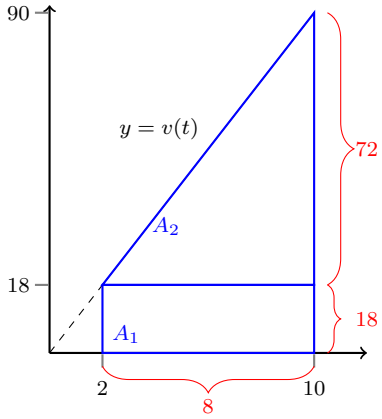
Solution: The main idea here is to turn this back into area:

$$\begin{aligned}\text{distance} &= \text{changing velocity} \times \text{time} \\ &= \text{changing height of } 9t \text{ graph} \times \text{width along graph} \\ &= \text{area under "curve"}\end{aligned}$$

So, let's draw this graph, and then calculate the distance under it (more precisely, between the graph and the horizontal axis, and between $t = 2$ and $t = 10$)



Maybe you know the area formula for a trapezoid, but if not, it's easy to break this shape up into a rectangle and a triangle:



Now we can finish our calculation:

$$\text{distance} = \text{area}$$

$$
\begin{aligned}
&= A_1 + A_2 \\
&= 8(18) + \frac{1}{2}(8)(72) \\
&= 432
\end{aligned}
$$

5.2 The Definite Integral

5.2 Example 1

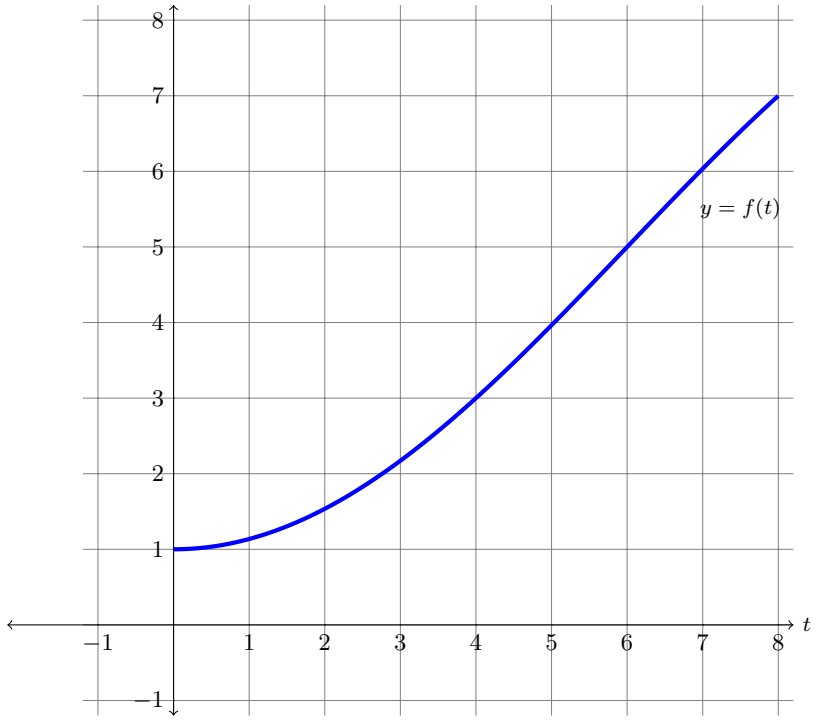
Using the graph of $f(t)$ below, draw rectangles representing each of the following Riemann sums for the function $f(t)$ on the interval $0 \leq t \leq 8$ (or $t \in [0, 8]$). Calculate the value of each sum.

(a) Left-hand sum with $\Delta t = 4$ ($n =$?)

(b) Right-hand sum with $\Delta t = 4$ ($n =$?)

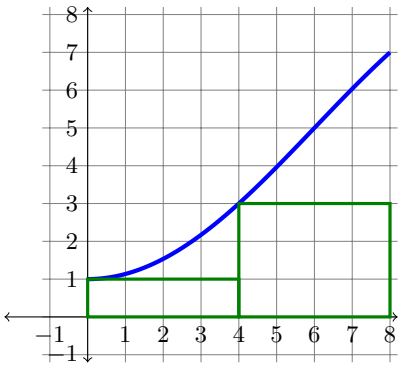
(c) Left-hand sum with $\Delta t = 2$ ($n =$?)

(d) Right-hand sum with $\Delta t = 2$ ($n =$?)



Solution:

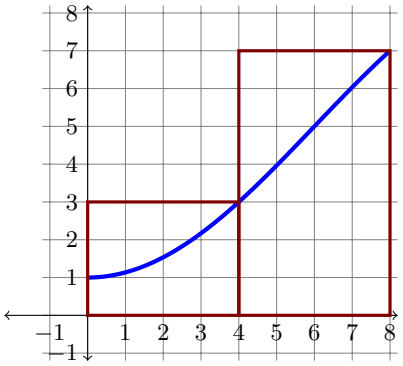
(a) Since $\Delta t = 4$, and the total interval has length 8, this means we divided it into two pieces. Therefore, we'll draw two rectangles, one from 0 to 4 and the other from 4 to 8. Each rectangle gets its height from the left edge. So the first one will get its height from the point on the curve where $x = 0$, and the second one will get its height from the point on the curve where $x = 4$. The picture looks like this



and we get

$$\begin{aligned}\text{Riemann Sum} &= \text{sum of areas of two rectangles}\\ &= f(0)\cdot\Delta t + f(4)\cdot\Delta t\\ &= 1\cdot 4 + 3\dot{4}\\ &= 4+12\\ &= 16\end{aligned}$$

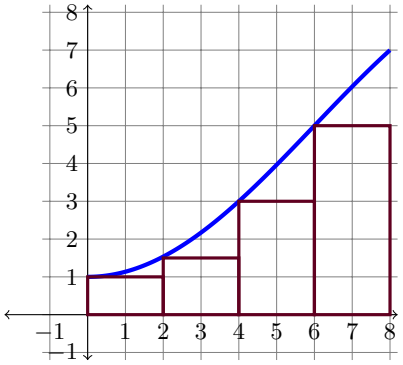
(b) We'll be briefer this time: $\Delta t = 4$, $n = 2$, each rectangle gets its height from the right edge, the first one uses $x = 4$ and the second one uses $x = 8$:



and we get

$$\begin{aligned}\text{Riemann Sum} &= \text{sum of areas of two rectangles}\\ &= f(4)\cdot\Delta t + f(8)\cdot\Delta t\\ &= 3\cdot 4 + 7\cdot 4\\ &= 12+28\\ &= 40\end{aligned}$$

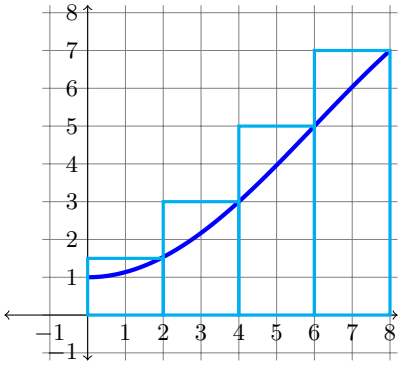
(c) $\Delta t = 2$, $n = 4$, heights from the left edge, use $x = 0$, 2, 4, 6:



and we get

$$\begin{aligned}\text{Riemann Sum} &= \text{sum of areas of four rectangles} \\ &= f(0)\cdot\Delta t + f(2)\cdot\Delta t + f(4)\cdot\Delta t + f(6)\cdot\Delta t \\ &= 1\cdot 2 + 1.5\cdot 2 + 3\cdot 2 + 5\cdot 2 \qquad\qquad = 2+3+6+10 \\ &= 21\end{aligned}$$

(d) $\Delta t = 2$, $n = 4$, heights from the right edge, use $x = 2$, 4, 6, 8:



and we get

$$\begin{aligned}\text{Riemann Sum} &= \text{sum of areas of four rectangles} \\ &= f(2)\cdot\Delta t + f(4)\cdot\Delta t + f(6)\cdot\Delta t + f(8)\cdot\Delta t \\ &= 1.5\cdot 2 + 3\cdot 2 + 5\cdot 2 + 7\cdot 2 \\ &= 33\end{aligned}$$

5.3 The Definite Integral as Area

5.3 Example 1: Problem 7

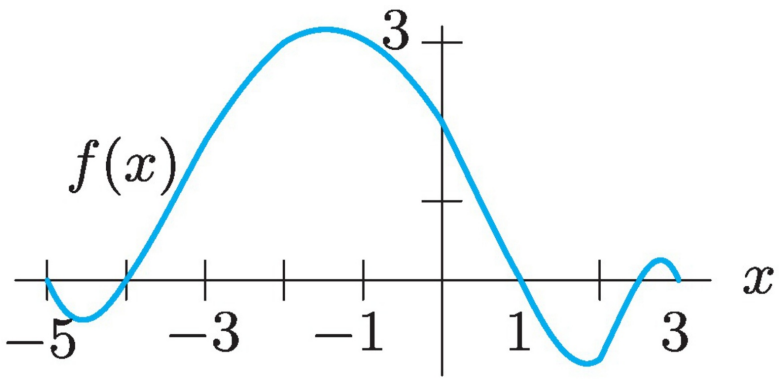
Using the figure below (Figure 5.36 in the text), decide whether each of the following definite integrals is positive or negative.

(a) $\int_{-5}^{-4} f(x)\,dx$

(b) $\int_{-4}^{1} f(x)\,dx$

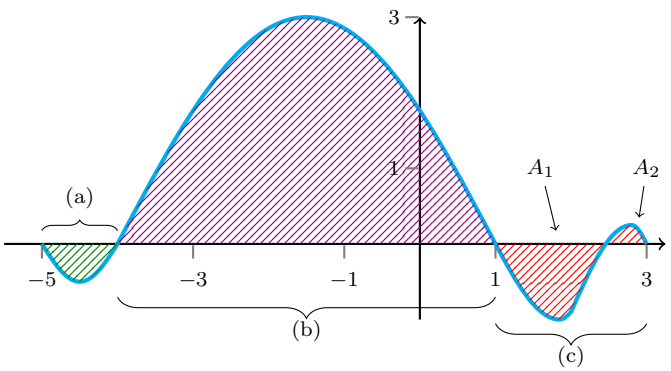
(c) $\int_{1}^{3} f(x)\,dx$

(d) $\int_{-5}^{3} f(x)\,dx$



Solution: The main idea of this example is simply to translate integrals into net area. So, $\int_{-5}^{-4} f(x)\,dx$ is the net area, between $f(x)$ and the x-axis, from $x = -5$ to $x = -4$. We can do some parts of this problem without coloring anything in. For instance, in part (a), the graph of $f(x)$ is entirely below the x-axis, and therefore the integral is negative.

But, sooner or later it's useful to have different parts shaded in, so I'll go ahead and do that



(a) $\int_{-5}^{-4} f(x)\,dx$ is area below the x-axis, as labeled above, so it's negative.

(b) $\int_{-4}^{1} f(x)\,dx$ is positive.

(c) $\int_{1}^{3} f(x)\,dx = A_2 - A_1$. It's pretty easy to see that A_1 is bigger, and so this integral, i.e. the net area, is negative.

(d) $\int_{-5}^{3} f(x)\,dx$ = (a) + (b) + (c). Although (a) and (c) are negative, it's pretty easy to see that (b) is larger than both (a) and (c) combined (actually, to be more accurate I should say that (b) is larger than both the absolute values of (a) and (c) combined). So, the net result is positive.

5.3 Example 2

The following graph shows the function f. Evaluate the integrals.

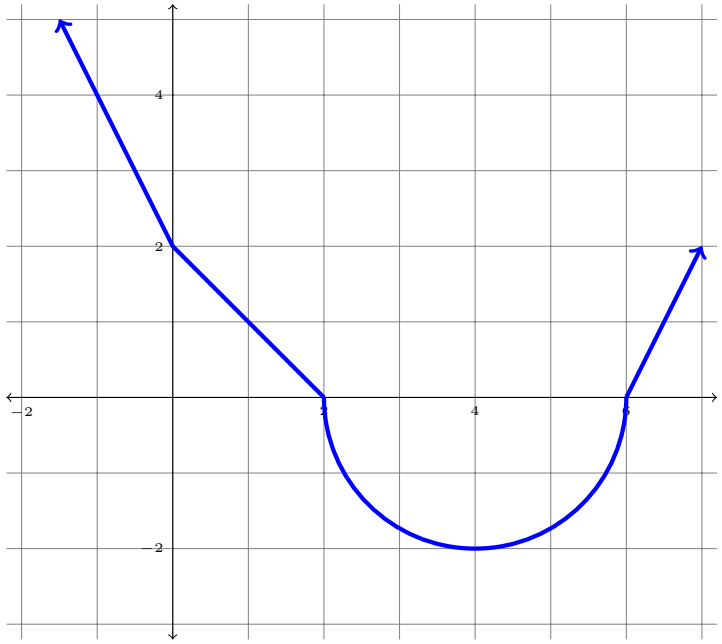
(a) $\int_{-1}^{0} f(x)\,dx$

(b) $\int_{0}^{2} f(x)\,dx$

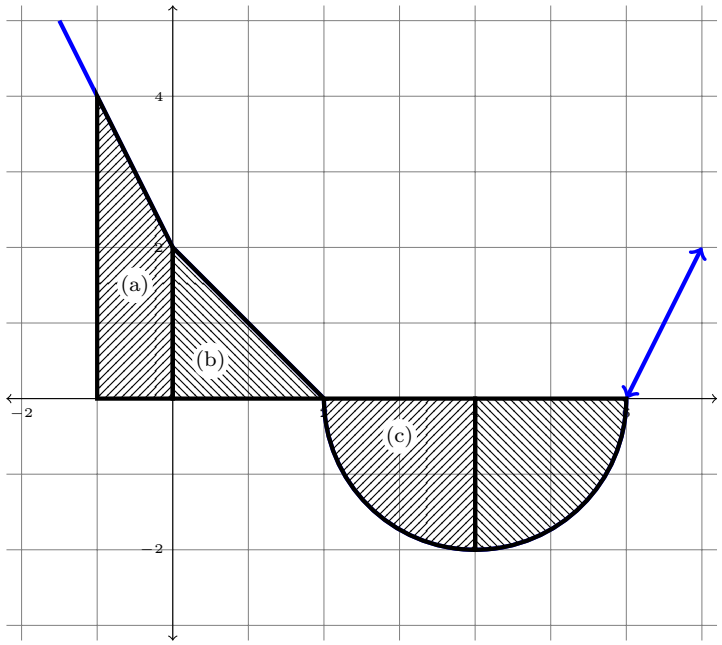
(c) $\int_{2}^{4} f(x)\,dx$

(d) $\int_{0}^{4} f(x)\,dx$

(e) $\int_{0}^{6} f(x)\,dx$



Solution: This example is similar to the previous example, the only difference is that in the last example we could just estimate what was positive and what was negative, but here we can calculate the exact integrals using basic geometric formulas for area. We'll start by dividing the regions, shading them and labeling them.



(a) This is labeled with (a) above. It's a triangle plus a rectangle, $\frac{1}{2}(1)(2) + (1)(2) = 3$.

(b) This is labeled with (b) above. It's a triangle, $\frac{1}{2}(2)(2) = 2$.

(c) This is labeled with (c) above. It's one quarter of a circle with radius 2. Since it's below the x-axis it should be negative, so $\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2^2) = -\pi$.

(d) $\int_0^4 f(x)\,dx =$(c) + another quarter circle. Thus, it's -2π.

(e) $\int_0^6 f(x)\,dx =$(b) + (d) $= 2 - 2\pi$.

5.4 Interpretations of the Definite Integral

5.4 Example 1

Explain in words what each integral represents and give the units

(a) $v(t)$ is velocity in mph and t is time in hours,

$$I = \int_2^5 v(t)\ dt.$$

(b) $a(t)$ is acceleration in m/s^2 and t is in seconds,

$$I = \int_3^4 a(t)\ dt.$$

(c) $f(t)$ is the rate at which water is flowing out of a water main break in liters per seconds, and t is in seconds,

$$I = \int_0^3 f(t)\ dt.$$

Solution:

(a) To figure out what *any* integral is giving you, look at everything after the integral sign:

$$\begin{aligned} v(t)\,dt &\approx v \times \Delta t \\ &= \text{velocity} \times \Delta t \\ &= \frac{\Delta\text{position}}{\Delta t} \times \Delta t \\ &= \Delta\text{position} \end{aligned}$$

To state it more completely: this is the net change in position from 2 to 5 hours.

To figure out the units, simply multiply the units of the above quantities:

$$\begin{aligned} \text{units} &= \text{units of } v \times \text{units of } t \\ &= \frac{\text{miles}}{\text{hour}} \times \text{hours} \\ &= \text{miles} \end{aligned}$$

(b)

$$\begin{aligned} a(t)\,dt &\approx a \times \Delta t \\ &= \text{acceleration} \times \Delta t \\ &= \frac{\Delta\text{velocity}}{\Delta t} \times \Delta t \end{aligned}$$

$$= \Delta\text{velocity}$$

To state it more completely: this is the net change in velocity from 3 to 4 seconds.

To figure out the units, simply multiply the units of the above quantities:

$$\begin{aligned}\text{units} &= \text{units of } a \times \text{units of } t \\ &= \frac{\text{meters}}{\text{second}^2} \times \text{seconds} \\ &= \text{meters/second}\end{aligned}$$

(c)

$$\begin{aligned}f(t)\,dt &\approx f \times \Delta t \\ &= \text{rate of water flow} \times \Delta t \\ &= \frac{\Delta\text{volume of water}}{\Delta t} \times \Delta t \\ &= \Delta\text{volume of water}\end{aligned}$$

To state it more completely: this is the net change in velocity from 0 to 3 seconds.

To figure out the units, simply multiply the units of the above quantities:

$$\begin{aligned}\text{units} &= \text{units of } f \times \text{units of } t \\ &= \frac{\text{liters}}{\text{second}} \times \text{seconds} \\ &= \text{liters}\end{aligned}$$

5.5 Total Change and the Fundamental Theorem of Calculus

5.5 Example 1

The marginal cost $C'(q)$ of making T-shirts is shown below. Suppose the fixed cost is \$100.

q	0	20	40	60	80	100
$C'(q)$	10	4.67	3.95	3.58	3.33	3.15

(a) Estimate the total cost of making 60 T-shirts.

(b) What is the total variable cost of making 60 T-shirts?

(c) Estimate the difference in cost between making 60 T-shirts and 100.

Solution:

(a)

$$\begin{aligned}\text{cost in making 60 T-shirts} &= C(60)\\ &= \text{fixed cost } + \text{variable cost}\\ &= C(0) + \int_0^{60} C'(q)\,dq\end{aligned}$$

Now we estimate this integral, using both Left Hand Sums and Right Hand Sums.

$$\begin{aligned}\int_0^{60} C'(q)\,dq &\approx 10(20) + 4.67(20) + 3.95(20) \qquad \text{(LHS)}\\ &= 372.40\\ \int_0^{60} C'(q)\,dq &\approx 4.67(20) + 3.95(20) + 3.58(20) \qquad \text{(RHS)}\\ &= 244\end{aligned}$$

Averaging these gets us

$$\begin{aligned}\int_0^{60} C'(q)\,dq &\approx 308.20\\ C(60) &\approx 100 + 308.20\\ &= \$408.20\end{aligned}$$

(b) The total variable cost is the part of our previous answer that *didn't* come from fixed cost. In other words

$$\$308.20$$

(c)

$$\text{Change in cost from 60 to 100} = \int_{60}^{100} C'(q)\,dq$$

$$\text{Left-sum: } \int_{60}^{100} C'(q)\,dq \approx 3.58(20) + 3.33(20)$$
$$= \$138.20$$
$$\text{Right-sum: } \int_{60}^{100} C'(q)\,dq \approx 3.33(20) + 3.15(20)$$
$$= \$129.6$$
$$\text{average: } \int_{60}^{100} C'(q)\,dq \approx \$133.90$$

5.5 Example 2

A cup of coffee is put into a 70 °F room when $t = 0$. The temperature (in °F) of the coffee t minutes after being in the room is given by

$$H(t) = 110e^{-0.1672t} + 70.$$

(a) Find $H'(t)$ and explain in words what this represents.

(b) What is $H(0)$ and what does it represent?

(c) What does $\int_2^4 H'(t)\,dt$ represent, and what is that value?

(d) How much does the temperature change in the first 5 minutes in the room?

Solution:

(a) Do you still remember how to take derivatives? Basically, we need the Chain Rule here for $e^{-0.1672t}$, so at the right step we will multiply by the derivative of what's on top:

$$H'(t) = 110(-0.1672)e^{-0.1672t} = -18.392e^{-0.1627t}$$

$H'(t)$ gives the rate the coffee's temperature is changing.

(b) $H(0) = 110 + 70 = 180$. This is the temperature of the coffee when it enters the room.

(c) $\int_2^4 H'(t)\,dt$ represents how much the temperature changes between the 2nd and 4th minutes in the room.

$$\begin{aligned}\int_2^4 H'(t)\,dt &= H(4) - H(2)\\ &= \left(110e^{-0.1672(4)} + 70\right) - \left(110e^{-0.1672(2)} + 70\right)\\ &\approx -22.3789\end{aligned}$$

So it decreases by about 22.389 °F.

(d)

$$\begin{aligned}\int_0^5 H'(t)\,dt &= H(5) - H(0)\\ &= 110e^{-0.1672(5)} + 70 - 180\\ &\approx -62.3215\end{aligned}$$

So the temperature of the coffee decreases by about 62 °F. The important point here is that you shouldn't wait this long to start drinking your coffee: the best temperature is around 140 degrees and it's cooled too much!